Resolution of Itô Calculus and the Black-Scholes Formula

1. The Fundamental Result

The following is the Itô formula, which lies at the heart of what is to follow. It should be interpreted as an integral version of the chain rule for Itô processes.

Theorem. *Suppose that X_t is an Itô process defined by $X_T = X_0 + \int_0^T f(t)\, dB_t + \int_0^T g(t)\, dt$ and that $F(t,x)$ is a sufficiently smooth function of two variables. Then*

$$\begin{aligned} F(T, X_T) - F(0, X_0) &= \int_0^T F_t(t, X_t)\, dt + \int_0^T F_x(t, X_t) f(t)\, dB_t \\ &+ \int_0^T F_x(t, X_t) g(t)\, dt + \frac{1}{2} \int_0^T F_{xx}(t, X_t) f^2(t)\, dt \end{aligned} \tag{1}$$

2. The Setup

2.1. The Model for Equity Prices

We first recall the model we are using for equity prices. Let B_t be standard Brownian motion and define an Itô process X_t by

$$X_t = X_0 + \sigma B_t + \mu t. \tag{2}$$

From the Itô formula, we obtain the following:

Theorem. *Let $Y_t = e^{X_t}$. Then Y_t is an Itô process and if $\alpha = \mu + \frac{1}{2}\sigma^2$, then*

$$Y_T - Y_0 = \int_0^T Y_t \sigma\, dB_t + \int_0^T Y_t \alpha\, dt. \tag{3}$$

Armed with this result, we can now use our definition of an integral with respect to an Itô process to model the capital gains in a stock position. More precisely, we will say that the capital gains from holding $\Delta(t)$ shares of a stock priced by Y_t at time from time 0 until time T will yield a capital gain of

$$\int_0^T \Delta(t)\, dY_t. \tag{4}$$

2.2. A Hedging Portfolio

We can use the above model to valuate a portfolio which holds a position in an equity and invests the rest in a riskfree money market fund which yields a rate of return equal to r. So assume that the value of the entire portfolio at time t is Z_t, and that

- at time t, the portfolio contains $\Delta(t)$ shares of an equity priced at Y_t, and
- the remainder of the portfolio's assets are held in a riskfree money market account which yields an interest rate of r.

The value of the equity portion at time t is $\Delta(t) Y_t$ and the value of the money market holdings is $Z_t - \Delta(t) Y_t$. Thus the capital gain of the entire portfolio from time 0 to time T is

$$Z_T - Z_0 = \int_0^T \Delta(t)\, dY_t + \int_0^T r(Z_t - \Delta(t) Y_t)\, dt \tag{5}$$

By using the definition of an integral with respect to an Itô process, this expression can be rewritten as

$$
\begin{aligned}
Z_T - Z_0 &= \int_0^T \Delta(t)\, dY_t + \int_0^T r(Z_t - \Delta(t)Y_t)\, dt \\
&= \int_0^T \Delta(t)Y_t\sigma\, dB_t + \int_0^T \Delta(t)Y_t\alpha\, dt + \int_0^T r(Z_t - \Delta(t)Y_t)\, dt \\
&= \int_0^T rZ_t\, dt + \int_0^T \Delta(t)(\alpha - r)Y_t\, dt + \int_0^T \Delta(t)\sigma Y_t\, dB_t. \qquad (6)
\end{aligned}
$$

The final expression divides the capital gain into three components; the first shows the average underlying rate of return if the portfolio assets were held without risk, the second is the risk premium, and the last accounts for the volatility of the portfolio. Furthermore, if we combine the first two integrals, we obtain an expression for Z_t as an Itô process.

3. The Main Question

Consider a European call option with the following parameters:

- the call option is for an equity priced by the Itô process Y_t, and
- the strike price, that is, the right to buy one share of stock at time $t = T$, is K.

The value of the call option when $t = T$ is easy to figure out. It is simply $(Y_T - K)^+$. Of course, one is really interested in pricing this call option when $t < T$. Following the work of Black and Scholes, we assume that the price of the call option should depend only on:

- the time interval $T - t$,
- the value of the underlying stock Y_t,
- r, the interest rate,
- σ, the volatility,
- K, the strike price.

Note that the only variables in this list are the stock price and time. Let $c(t, x)$ be the value of the call at time t and stock price x.

Question. *What is the function $c(t, x)$? Or more explicitly, given a stock price x and time t, what should be the price of the call option with the above parameters?*

4. The Economics Principles

The fundamental insight of Fischer Black and Myron Scholes into pricing an option was to realize that a portfolio could be created which, with an appropriate trading strategy, would exactly mimic the price of the call option. More mathematically, for an appropriate position $\Delta(t)$, we would have

$$Z_t = c(t, Y_t)$$

for all values of t. This would allow the person who sells the call option to perfectly hedge his position by selling *short* the same portfolio. The mathematical consequences of this observation allow us to find $c(t, x)$. In practice, we will examine the discounted equation

$$e^{-rt} Z_t = e^{-rt} c(t, Y_t)$$

5. The Mathematics

Using the Itô formula, we will compute the discounted option price and the discounted value of the portfolio we use to hedge it.

5.1. Discounted option price

According to the Itô formula, we can let $F(t,x) = c(t,x)$ and conclude that

$$\begin{aligned} c(T,Y_T) - c(0,Y_0) &= \int_0^T c_t(t,Y_t)\,dt + \int_0^T c_x(t,Y_t)\,dY_t + \frac{1}{2}\int_0^T c_{xx}(t,Y_t)(Y_t\sigma)^2\,dt \\ &= \int_0^T \big(c_t(t,Y_t) + \alpha Y_t c_x(t,Y_t) + \frac{1}{2}c_{xx}(t,Y_t)(Y_t\sigma)^2\big)\,dt \\ &+ \int_0^T \sigma Y_t c_x(t,Y_t)\,dB_t. \end{aligned} \tag{7}$$

Furthermore, if we let $F(t,x) = e^{-rt}c(t,x)$ we can use the same formula to compute the discounted option price:

$$\begin{aligned} e^{-rT}c(T,Y_T) - c(0,Y_0) &= \int_0^T -re^{-rt}c(t,Y_t)\,dt + \int_0^T e^{-rt}c_t(t,Y_t)\,dt \\ &+ \int_0^T e^{-rt}c_x(t,Y_t)\sigma Y_t\,dB_t + \int_0^T e^{-rt}c_x(t,Y_t)\alpha Y_t\,dt \\ &+ \frac{1}{2}\int_0^T e^{-rt}c_{xx}(t,Y_t)(\sigma Y_t)^2\,dt \\ &= \int_0^T e^{-rt}\Big(-rc(t,Y_t) + c_t(t,Y_t) + c_x(t,Y_t)\alpha Y_t + \frac{1}{2}c_{xx}(t,Y_t)(\sigma Y_t)^2\Big)\,dt \\ &+ \int_0^T e^{-rt}\sigma Y_t c_x(t,Y_t)\,dB_t. \end{aligned} \tag{8}$$

5.2. Discounted portfolio price

We first use the Itô formula to compute the evolution of the discounted stock price $e^{-rt}Y_t$.

$$\begin{aligned} e^{-rT}Y_T - Y_0 &= \int_0^T -re^{-rt}Y_t\,dt + \int_0^T e^{-rt}\,dY_t \\ &= \int_0^T (\alpha - r)e^{-rt}Y_t\,dt + \int_0^T e^{-rt}\sigma Y_t\,dB_t. \end{aligned} \tag{9}$$

Furthermore, we can perform a similar computation for the discounted portfolio value. This is not hard, but you must remember our expression for the portfolio value Z_t as an Itô process.

$$\begin{aligned} e^{-rT}Z_T - Z_0 &= \int_0^T -re^{-rt}Z_t\,dt + \int_0^T e^{-rt}\Delta(t)\sigma Y_t\,dB_t \\ &+ \int_0^T e^{-rt}\Big(rZ_t + \Delta(t)(\alpha - r)Y_t\Big)\,dt \\ &= \int_0^T e^{-rt}\Delta(t)\sigma Y_t\,dB_t + \int_0^T e^{-rt}\Delta(t)(\alpha - r)Y_t\,dt \end{aligned} \tag{10}$$

5.3. Equating the Evolutions

We would like to make sure that our hedging portfolio Z_t is exactly equal to the option price $c(t,Y_t)$ at all points in time. This will happen if and only if

$$e^{-rt}Z_t = e^{-rt}c(t,Y_t)$$

for all values of $t \in [0,T)$. Furthermore, if we make sure that $Z_0 = c(0,Y_0)$, then we can use our formulas from above. More explicitly, for Z_t to equal $c(t,Y_t)$ for all $t \in [0,T)$, we must have

$$\int_0^t e^{-rs}\Big(-rc(s,Y_s) + c_t(s,Y_s) + c_x(s,Y_s)\alpha Y_s + \frac{1}{2}c_{xx}(s,Y_s)(\sigma Y_s)^2\Big)\, ds \tag{11}$$
$$+\int_0^t e^{-rs}\sigma Y_s c_x(s,Y_s)\, dB_s = \int_0^t e^{-rs}\Delta(s)\sigma Y_s\, dB_s + \int_0^t e^{-rs}\Delta(s)(\alpha - r)Y_s\, ds$$

This yields two equations:

$$\Delta(t)\sigma Y_t = c_x(t,Y_t)\sigma Y_t \tag{12}$$
$$(\alpha - r)Y_t\Delta(t) = -rc(t,Y_t) + c_t(t,Y_t) + \alpha Y_t c_x(t,Y_t) + \frac{1}{2}\sigma^2 Y_t^2 c_{xx}(t,Y_t), \tag{13}$$

which can be simplified to

$$\Delta(t) = c_x(t,Y_t) \tag{14}$$
$$rc(t,Y_t) = c_t(t,Y_t) + rY_t c_x(t,Y_t) + \frac{1}{2}\sigma^2 Y_t^2 c_{xx}(t,Y_t) \tag{15}$$

The second equation, after the substitution $Y_t = x$, becomes the partial differential equation

$$c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} = rc$$

whose solution $c = c(t,x)$ satisfies the terminal condition $c(T,x) = (x-k)^+$. If the solution to this PDE is known, the conditions of the first equation are easily satisfied. In fact, if we knew the function $c(t,x)$, the equation tells us exactly how many shares of stock to hold at a given time t. This is called the *delta-hedging rule*, and more explicitly, it says that the number of shares $\Delta(t)$ held by the portfolio Z_t should be the partial derivative with respect to the stock price of the option value at that time.

Theorem. *Suppose that $c(t,x)$ is solution to the the partial differential equation*

$$c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} = rc$$

which satisfies the terminal condition $c(T,x) = (x-K)^+$. Furthermore, assume that the price of an equity is given by the Itô process Y_t, the portfolio Z_t is as described above, and that the number of shares of Y_t held at any point in time is $\Delta(t) = c_x(t,Y_t)$. If $Z_0 = c(0,Y_0)$ then $c(t,Y_t) = Z_t$ for all $t \in [0,T]$.

Proof. If a portfolio starts with $Z_0 = c(0,Y_0)$ and uses the formula $\Delta(t) = c_x(t,Y_t)$ to determine the size of his equity investment, then both equations above will be satisfied since $c(t,x)$ is a solution of the second. Using the Itô formula in reverse, we have that

$$e^{-rt}Z_t - Z_0 = e^{-rt}c(t,Y_t) - c(0,Y_0)$$

is true for all $t \in [0,T)$ and, furthermore, $Z_t = c(t,Y_t)$ for all $t \in [0,T)$. Since Z_t, and $c(t,Y_t)$ are continuous, this means that as t approaches T, their values must agree in the limit as well. Hence

$$Z_T = c(T,Y_T).$$

But this means that $Z_T = (Y_T - K)^+$ as desired, since $c(T,Y_T) = (Y_T - K)^+$ via the terminal condition on the solution $c(t,x)$. $\square$

5.4. A Solution to the Partial Differential Equation

The explicit solution to the equation for $c(t, x)$ is beyond the scope of these notes; however, armed with the right notation, it is easily stated. Let $N(y) = \int_{-\infty}^{y} e^{\frac{z^2}{2}}\, dz$ be the cumulative standard normal distribution and let

$$d_{\pm}(\tau, x) = \frac{1}{\sigma\sqrt{t}}\Big(\log \frac{x}{K} + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)$$

Then the price of a call option at time t when the underlying stock price is x is given by:

$$c(t, x) = xN(d_+(T - t, x)) - Ke^{-r(T-t)}N(d_-(T - t, x)).$$

Note that the solution does not depend on α!