Bowdoin College

Math 3603: Advanced Analysis
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Homework 9

1. We aim to define the space of square-integrable functions, but the definition is a little subtle. The purpose of this exercise is to flush out this sublety. Consider a measure space $(X, \mathcal{F}, \mu)$ and two functions $f, g : X \to \mathbb{R}$. We say that they are *equivalent* iff $f = g$ except perhaps on a set of measure zero. If we write $[g]$ for the equivalence class of functions containing g, then

$$\mathcal{L}^2(X) = \{[f] \mid \int_X f^2 \, d\mu < \infty\}.$$

When endowed with the metric ρ defined by $\rho(f, g) = \sqrt{\int_X (f-g)^2}$, the set $\mathcal{L}^2(X)$ becomes a complete metric space.

   (a) Verify that the relation $f \sim g$ iff $f = g$ a.e. is indeed an equivalence relation. Be precise.

   (b) Prove that $\mathcal{L}^2(X)$ is well-defined.

   (c) Prove that ρ is well-defined.

2. Consider a sequence Π_n of partitions of the interval $[a, b]$ whose mesh approaches zero as n increases, and suppose that $f : [a, b] \to \mathbb{R}$ is continuous. For each n, let $f_n : [a, b] \to \mathbb{R}$ be the step function constant on each interval of Π_n that agrees with f on the left endpoint of each such interval. Prove the following lemma, which will be crucial in our construction of the general stochastic integral:

   Lemma. *For every $\epsilon > 0$, there exists an integer N such that for all $k, l > N$, we have*

$$|f_k(t) - f_l(t)| < \epsilon.$$

3. To compute the quadratic variation of Brownian motion (an upcoming attraction), we will need a fact about normally-distributed random variables. Show that if f is a normally-distributed random variable with expected value 0 and variance σ^2, then

$$E(f^4) = 3(\sigma^2)^2.$$

Computing the expected value of a random variable with a particular distribution is harder than it seems, at least at first glance. If f is the random variable under consideration, the formula is simply

$$E(f) = \int_X f \, d\mu.$$

However, in many situations the probability space X is not clearly defined and actually carrying this computation out is impossible. However, suppose that we know that f is normally distributed. For simplicity, assume that its expected value is 0 and variance is σ^2, so the underlying probability density function is

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

More precisely, this means that $\mu(\{x \in X \mid f(x) \in (a,b)\} = \int_a^b F(x)\, dx$. It is this definition that must be applied which will enable us to compute $E(f)$, but the process is not simple. Recall that the Lebesgue integral is defined as a supremum of integrals of simple functions. In fact, by a result from class, we can find an increasing sequence of simple functions $\{s_n\}$ which converges to f. You should review the construction in your notes, but here are some details:

Let I_i be the interval $[\frac{i-1}{2^n}, \frac{i}{2^n})$ and let $E_i = f^{-1}(I_i)$. The value of the simple function s_n on E_i is $\frac{i-1}{2^n}$. Then, since $\mu(\{x \in X \mid f(x) \in (a,b)\} = \int_a^b F(x)\, dx$, we have

$$\mu(E_i) = \int_{I_i} F(x)\, dx.$$

Furthermore:

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X s_n \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mu(E_i)$$

We can now use our formula for $\mu(E_i)$ and continue:

$$\int_X f \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \int_{I_i} F(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n2^n} \int_{I_i} \frac{i-1}{2^n} F(x)\, dx$$

Since $I_i = [\frac{i-1}{2^n}, \frac{i}{2^n})$, $\frac{i-1}{2^n} \leq x$ for all $x \in I_i$, hence

$$\int_X f \, d\mu \leq \lim_{n\to\infty} \sum_{i=1}^{n2^n} \int_{I_i} xF(x)\, dx \leq \lim_{n\to\infty} \sum_{i=1}^{n2^n} \int_{I_i} \frac{i}{2^n} F(x)\, dx = \int_X f \, d\mu.$$

By the Sandwich Theorem, this means that

$$\int_X f \, d\mu = \lim_{n\to\infty} \sum_{i=1}^{n2^n} \int_{I_i} xF(x)\, dx = \lim_{n\to\infty} \int_{\cup I_i} xF(x)\, dx = \int_{\mathbb{R}} xF(x)\, dx.$$

This argument is easily adapted to a more general situation. Consequently, we have just verified:

Theorem. *Suppose f is a random variable with probability density function F. Then*

$$E(f) = \int_X f \, d\mu = \int_{\mathbb{R}} xF(x)\, dx.$$

Note that in the above example, where $F(x)$ is centered at the origin, $E(f) = 0$ since $F(x)$ is symmetric about the origin and therefore $xF(x)$ is an odd function.