Bowdoin College

Math 3603: Advanced Analysis
Prof. Thomas Pietraho
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Homework 8

1. The following sequence of exercises concerns the notion of quadratic variation of a function. It is a notion absent from regular calculus and analysis, since the functions they usually consider have quadratic variation equal to zero. This will not be the case in the functions which will interest us in stochastic calculus.

   Definition. *Consider a random variable f defined on an interval $[a,b] \subset \mathbb{R}$. The quadratic variation of f on $[a,b]$ is defined as*

   $$[f,f]([a,b]) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} (f(t_{i+1}) - f(t_i))^2$$

   where Π is a partition $a = t_0 < t_1 < \ldots t_n = b$ and $||\Pi||$ denotes the mesh of Π.

   In stochastic calculus, the interval $[a,b]$ is usually of the form $[0,T]$ and $[f,f]([0,T])$ is written as $[f,f](T)$ and called the *variation of f up to time T.* When a function is defined only on a finite set of values, i.e. $f : \{t_1, t_2, \ldots, t_m\} \to \mathbb{R}$, quadratic variation is defined with respect to the counting measure and the resulting formula is

   $$[f,f]\{t_1, t_2, \ldots, t_m\} = \sum_{i=1}^{m-1} (f(t_{i+1}) - f(t_i))^2$$

   This version of quadratic variation is useful for discrete processes like a random walk, while the first version is useful for continuous ones, like Brownian motion (which we are yet to discuss).

   Problem: Show that a continuous function f which has a continuous derivative must have zero quadratic variation for any interval $[a,b]$. This is the fact alluded to above: functions of interest in calculus have "uninteresting" quadratic variation.

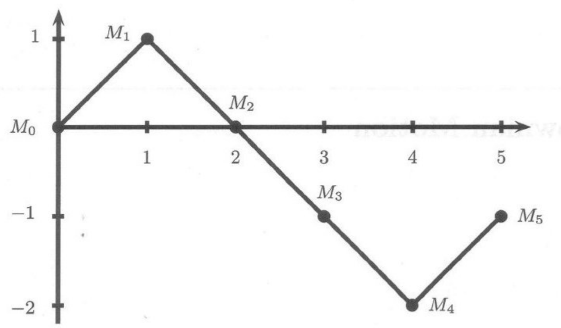
   Hint: Use the Mean Value Theorem to show that

   $$[f,f]([a,b]) \leq \lim_{||\Pi|| \to 0} ||\Pi|| \cdot \int_a^b |f'(t)|^2 d\mu.$$

2. Consider a number $a \in (0,1]$ and interpret it as a sequence of coin flips by expressing it as a non-terminating binary decimal. Then a determines a random walk function via

   $$f_k(a) = \sum_{i=1}^{k} R_i(a)$$

   Put another way, $f_k(a)$ tells us how far from the origin the random walker is after k flips. For example, if $a = .01110\ldots$, then the random walk can be represented by the picture



with $f_0(a) = 0$, $f_1(a) = 1$, $f_2(a) = 0$, $f_3(a) = -1$, $f_4(a) = -2$, and $f_5(a) = -1$. Note that the notation is somewhat deceptive; once a is fixed, the independent variable in $f_k(a)$ is k, and we treat it as a function of k. Write $f_\cdot(a)$ for this function of k. Compute

$$[f_\cdot(a), f_\cdot(a)]\{1, 2, 3, \ldots, n\}.$$

In other words, find the quadratic variation of the first n steps of a random walk. Does it depend on a?

3. For $k \in \mathbb{N}$, Let $B^1_k(a)$ be the discrete-time stochastic process defined by

$$B^1_k(a) = \sum_{i=1}^{k} R_i(a)$$

where R_i is the ith Rademacher function and $a \in (0, 1]$. This is the random walk generated by the non-terminating binary decimal expansion of a. We shorten the time interval between steps, and define another discrete-time stochastic process

$$B^n_{k/n}(a) = \frac{1}{\sqrt{n}} B^1_k(a)$$

for $k \in \mathbb{N}$. Compute the quadratic variation of $B^n_\cdot(a)$ over the discrete interval $[k_1/n, k_1 + 1/n, \ldots, k_2/n]$. In other words, find

$$[B^n_\cdot(a), B^n_\cdot(a)]\left\{\frac{k_1}{n}, \frac{k_1+1}{n}, \ldots, \frac{k_2}{n}\right\}.$$

4. In our verification of existence for Brownian motion, we will use a crucial inequality without proof. As this is a state which cannot persist, the purpose of this exercise is to remedy the situation. Let f be a normally-distributed random variable with mean 0 and variance 1 defined on a probability space X. Let μ be probability measure on X. Show that whenever $a > 0$,

$$\mu\{x \mid f(x) > a\} \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{a^2}{2}}}{a}$$

Hint: You can express the value of the left side precisely using an integral.

5. For each positive integer n, define a function $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}.$$

This is the normal density function with mean zero and variance n.

(a) What is the function $f(x) = \lim_{n\to\infty} f_n(x)$? Stated more precisely, what is the pointwise limit of the sequence of functions $\{f_n(x)\}$?

(b) What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx$?

(c) Note that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx \neq \int_{-\infty}^{\infty} f(x)\,dx$$

Explain this fact in light of the Monotone Convergence Theorem.