# Bowdoin College

Math 3603: Advanced Analysis
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## Homework 7

1. Let the set of Bernoulli sequences be identified with the interval $(0, 1]$ in the usual way, and write $a_i(x)$ for the $i^{th}$ digit in the nonterminating binary expansion of a number $x \in (0, 1]$. The $i^{th}$ Rademacher function $R_i$ is defined by

$$R_i(x) = \begin{cases} 1 & \text{if } a_i(x) = 1, \text{ and} \\ -1 & \text{if } a_i(x) = 0. \end{cases}$$

Do the following exercises.

   (a) Show that the set $\{R_i\}$ is a family of independent, identically distributed random variables.

   (b) Compute the expected value and variance for each $R_i$.

   (c) Let $S_n = \sum_{i=1}^{n} R_i$, and compute the expected value and variance for each of the random variables $S_n - S_m$, where $m < n$. Under what circumstances are the random variables $S_n - S_m$ and $S_{n'} - S_{m'}$ independent?

2. The purpose of this exercise is to prove a general version of the weak law of large numbers. Consider a family $\{f_i\}$ of independent, identically distributed random variables, and let $E$ be their common expected value and $V$ be their common variance. Show that

$$\text{Prob}\Big(\Big|\frac{f_1 + f_2 + \dots f_n}{n} - E\Big| > \epsilon\Big) \leq \frac{V}{\epsilon^2 n}$$

for every $\epsilon > 0$. From this, conclude

**Theorem.** *Given $\epsilon > 0$ and a family $\{f_i\}$ of independent, identically distributed random variables with expected value $E$ and variance $V$, then*

$$\lim_{n\to\infty} \text{Prob}\Big(\Big|\frac{f_1 + f_2 + \dots f_n}{n} - E\Big| > \epsilon\Big) = 0.$$

3. Suppose that $f$ and $g$ are two random variables on the probability space $(X, \mathcal{F}, \mu)$. Show that

   (a) $E(f + g) = E(f) + E(g)$, and

   (b) $V(f + g) = V(f) + V(g)$ if $f$ and $g$ are independent.

   (c) $V(cf) = c^2 V(f)$ for $c \in \mathbb{R}$.