Bowdoin College

Math 3603: Advanced Analysis
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Homework 5

1. Consider the following definition:

   Definition. *We will say that $x \in \mathbb{R}$ is* **random** *if for every $n \in \mathbb{N}$, the number n appears as a sequence of consecutive digits in the decimal expansion of x.*

   No rational number is random, and some irrational numbers are not random. However, it is still an unsolved problem to decide whether the constant π is random. As a weak substitute, prove the following theorem:

   Theorem. *A real number $x \in I$ is random with probability one.*

2. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone increasing and let μ be Lebesgue measure. Show that f is measurable.

3. Let f_n be a sequence of measurable functions. Show that the set of points x in X where the sequence $\{f_n(x)\}$ converges to a finite number is measurable.

4. In class, we briefly discussed the notions of the limit supremum and limit infimum of a sequence of functions. The purpose of this exercise is to verify the details exploited in the lecture, albeit in the context of sequence of real numbers. We begin with the appropriate definitions:

   Definition. *Consider a sequence of real numbers $\{a_i\}$ and define another sequence of reals by*

   $$b_i = \inf_{k \geq i} a_k.$$

   Note that the new sequence $\{b_i\}$ is increasing, hence it has a supremum within the extended real numbers. Let $\liminf a_i = \sup_{i \in \mathbb{N}} b_i$.

   (a) Following the above, carefully define the notion of a $\limsup$ of a sequence of real numbers.

   (b) Explain why for any sequence, the $\limsup$ and $\liminf$ exist in $\widetilde{\mathbb{R}}$. Be as precise as you can.

   (c) Find a sequence of real numbers for which the $\limsup$ and $\liminf$ do not equal each other.

   (d) Finally, show that a sequence of real numbers has a limit iff its $\limsup$ and $\liminf$ coincide.