Bowdoin College

Math 3603: Advanced Analysis
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Homework 4

1. Let X be a set, $\mathcal{F}$ a σ-field of subsets of X, and μ a probability measure on $\mathcal{F}$. Let $\{A_i\}$ be a sequence of subsets of X belonging to $\mathcal{F}$. Show that if $A_1 \supset A_2 \supset A_3 \ldots$, then

$$\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i).$$

2. Let X be a set, $\mathcal{F}$ a σ-field of subsets of X, and μ a probability measure on $\mathcal{F}$. Consider a sequence of sets $\{A_i\} \in \mathcal{F}$. The purpose of this exercise is to detail the relationship between the sets $\{A_i, i.o.\}$ and $\{A_i, a.a.\}$. To that effect, show that

$$\liminf A_i^c = (\limsup A_i)^c.$$

   Conclude that $\mu(\liminf A_i^c) = 0$ iff $\mu(\limsup A_i) = 1$.

> **Definition:** Recall the map we established between Bernoulli sequences β and points in the unit interval I. If μ_L is Lebesgue measure and $\mathcal{M}$ is the family of all Lebesgue-measurable subsets of I, then $(I, \mathcal{M}, \mu_L)$ is a probability space. We will say that an event E occurring on Bernoulli sequences is **plausible** if the corresponding subset $B_E \in \mathcal{M}$.

3. Show that a gambler quadrupling his initial stake is a plausible event in the perpetual coin-tossing game where he wins one dollar if the coin flip is an H and loses a dollar if the coin flip is a T.

   Note: A gambler cannot quadruple his initial stake if he loses all his money beforehand.

4. Let N be a non-zero integer. Prove that a random walk on the line starting at zero passes either through the point N or the point $-N$ with probability one. Conclude that it passes through N with probability at least one-half.

5. The following describes the notion of a random walk without pauses on the real line. Starting at the origin, move left or right one unit based on the result of a coin toss; you will move to the left if it is a tail, and right if it is a head. Show that the event of visiting the origin infinitely often is plausible.