Bowdoin College

Math 3603: Advanced Analysis
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Homework 3

1. Recall that a set T is dense in a metric space X if there is a sequence of elements of T converging to every element of X. The purpose of this exercise is to show that there can a considerable difference between the measures of a set and a dense subset therein.

   > Consider the metric space $X = \mathbb{R}$, let $\mathcal{M}_X$ be the collection of Lebesgue-measurable subsets of $\mathbb{R}$ with measure μ_L. Show that for any $\delta > 0$ there is an open dense subset U of $\mathbb{R}$ with $\mu_L(U) < \delta$.

2. This exercise shows that Lebesgue measure on $\mathbb{R}$ is *translation invariant*. Consider $c \in \mathbb{R}$. Given any subset A of $\mathbb{R}$, define

   $$A + c = \{x \in \mathbb{R} \mid x - c \in A\}.$$

   This is just a translate of the set A by c. Prove that if A is Lebesgue-measurable, then $A + c$ is also Lebesgue-measurable and

   $$\mu_L(A + c) = \mu_L(A).$$

   Hint: First show this is true for intervals and then verify that the equation holds for outer measure.