Bowdoin College

Math 3603: Advanced Analysis
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Homework 2

1. Let X be the set of real numbers and let $\mathcal{R} = \mathcal{R}_{Leb}$ be the collection of finite unions of intervals. Define a set function μ on $\mathcal{R}$ by $\mu(A) = 1$ if A contains an interval of the form $(0, \epsilon)$ for some $\epsilon > 0$, and zero otherwise. Show that μ is an additive function which is not countably additive.

2. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous monotone increasing function. For an interval A with endpoints a and b, with $a, b \in \mathbb{R}$, let
$$\mu_F(A) = F(b) - F(a).$$
Consider a family of finite intervals $\{A_i\}$. We can extend this definition to sets that are finite unions of intervals by letting
$$\mu_F(\bigsqcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} \mu_F(A_i).$$
Show that μ_F is a measure on $\mathcal{R}_{Leb}$, that is, prove that it is countably additive.

3. Let X be an uncountable set. Define $\mathcal{R}$ to be the collection of all finite subsets of X. Given $A \in \mathcal{R}$ let $\mu(A)$ be the number of elements in A.
   (a) Show that $\mathcal{R}$ is a ring and that μ is a measure on $\mathcal{R}$.
   (b) Describe the corresponding outer measure μ^*.
   (c) What is $\overline{\mathcal{R}}$?

4. Let X be a set, $\mathcal{R}$ be a ring of subsets of X and μ a measure on $\mathcal{R}$. Let μ^* be the corresponding outer measure. It is possible but a bit tedious to prove the following:

   Theorem: If $A \in \mathcal{R}$, then for every $E \subset X$,
$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E).$$
   Show the following corollary of this result:

   Corollary: If $A \in \overline{\mathcal{R}}$, then for every $E \subset X$,
$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E).$$

5. Recall that $A \in \overline{\mathcal{R}}$ iff $A = \lim_{n\to\infty} A_n$ with each $A_n \in \mathcal{R}$. Show that
   (a) the Cantor set, and
   (b) the rational numbers,

   lie in $\overline{\mathcal{R}}$ when $X = \mathbb{R}$ and $\mathcal{R} = \mathcal{R}_{Leb}$ by finding a sequence of finite unions of intervals that converges to each one.