Bowdoin College

Math 3603: Advanced Analysis
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Homework 12

1. We defined the Itô stochastic integral as the *mean square limit*, that is the limit in the $L^2(X)$ metric space, of partial sums

$$\int_a^b f(t)\, dB_t = \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i)(B_{t_{i+1}} - B_{t_i})$$

for a partition $\Pi : a = t_0 < t_1 < \dots t_n = b$. This approach uses the left endpoint of each interval. A more general approach would be to choose an intermediate point $\tau_i \in [t_i, t_{i+1}]$ and define a general stochastic integral as

$$\int_a^b f(t)\, dB_t = \lim_{n\to\infty} \sum_{i=0}^{n} f(\tau_i)(B_{t_{i+1}} - B_{t_i})$$

So pick $\alpha \in [0,1]$, let $\tau_i = \alpha t_i + (1-\alpha)t_{i+1}$, and compute

$$E\Big(\int_0^T B_t\, dB_t \Big)$$

using the more general definition. What is the answer for the Itô integral, and how does the answer vary for $\alpha \in [0,1]$? From a finance perspective, why is the value $\alpha = 1$ used for the Itô integral?

Hint: Note that

$$B_{t_{i+1}} - B_{t_i} = B_{t_{i+1}} - B_{\tau_i} + B_{\tau_i} - B_{t_i}$$

and use the formula for covariance of Brownian motion.

2. This problem is aimed to supplement the proof of the Itô formula from class. Under the conditions of the theorem, show that with probability one,

$$\lim_{||\Pi||\to 0} \sum_{i=0}^{n-1} f_{xx}(t_i, B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 = \int_0^T f_{xx}(t, B_t)\, dt.$$

Hint: Use our formula for the quadratic variation of Brownian motion. Unlike our proof in class, be careful and make sure your manipulations are justified. One approach is to compute the expected value of both sides, and then the variance of their difference. You may assume the following are true in our current circumstances without proof:

- If $f(t_i, B_{t_i})$ and $(B_{t_{i+1}} - B_{t_i})$ are independent, then so are $f_{xx}(t_i, B_{t_i})$ and $(B_{t_{i+1}} - B_{t_i})$.
- Limit and integral operators commute.