Bowdoin College

Math 3603: Advanced Analysis
Prof. Thomas Pietraho
Spring, 2022

Homework 11

1. Suppose that $f, g : [a, b] \to \mathbb{R}$, f is continuous, and g is differentiable with continuous derivative. Historically, the following generalization of the Riemann-integral was a precursor to both the Lebesgue and stochastic integrals:

   Definition. *The Riemann–-Stieltjes integral is defined as*

   $$\int_a^b f(t)\, dg(t) = \int_a^b f(t) \cdot g'(t)\; dt$$

   where the right side is the usual Riemann integral.

   In class we will compute the stochastic integral $\int_a^b B_t\; dB_t$ for the non-differentiable Brownian motion B_t. By way of comparison, assume that $g(t)$ is differentiable on $[a, b]$ with continuous derivative and compute

   $$\int_a^b g(t)\; dg(t) = \int_a^b g(t) \cdot g'(t)\; dt.$$

2. The Brownian motion model for stock prices is too simple for our purposes; in particular, it implies no drift in the long term behavior of equities. The notion of an Itô process, defined as

   $$X_t = X_0 + \int_0^t f(x)\; dB_s + \int_0^t g(s)\; ds$$

   accounts for a much larger family of random motion. We defined the integral of an Itô process as follows:

   Definition. *Consider and Itô process X^t and an unanticipating random function $h(t)$ which satisfies*

   $$\int_0^t (hf)^2 < \infty \text{ and } \int_0^t |hg| < \infty.$$

   Then

   $$\int_0^t h(s)\; dX_s = \int_0^t h(s) f(s)\; dB_s + \int_0^t h(s) g(s)\; ds.$$

   If $h(t)$ is interpreted as the number of shares of a stock held at time t, explain how the above creates a model for capital gains if X_t is our model for the per-share price of the stock.