BOWDOIN COLLEGE

MATH 3603: Advanced analysis Prof. Thomas Pietraho

Homework 10

1. In class, we defined the first variation of a function over an interval [a, b] as

$$FV(f)[a,b] = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

where Π is a partition of [a, b] with breakpoints $\{t_i\}$. It computes the distance traveled by a particle whose position at time t is given by f(t). Show that if f is a function which is differentiable everywhere and whose derivative is continuous, then

$$FV(f)[a,b] = \int_a^b |f'(t)| dt.$$

Hint: Use the Mean Value Theorem.

2. We have the following theorem about functions of finite first variation from class:

Theorem. Suppose $f : [a,b] \to \mathbb{R}$ has $FV(f)[a,b] < \infty$. Then there exist monotone increasing functions g and h such that

$$f = g - h.$$

Our proof was constructive, meaning we actually found functions g and h that work. However, we had a lot of choice as this decomposition of f is not unique. If c is a constant, both g + c and h + c are monotone increasing and their difference is still equal to f. The following question suggests itself: Is the decomposition unique up to this constant? Prove it if so, and provide a counterexample otherwise.

- 3. Suppose that B_t is standard Brownian motion and define $X_t = X_0 + \mu t + \sigma B_t$ for constants X_0 , μ , and σ . Further, let $Y_t = e^{X_t}$. The stochastic process X_t is called (μ, σ^2) -Brownian motion and Y_t is the corresponding geometric Brownian motion.
 - (a) Find the expected value and variance of the increments of X_t . Are they independent?
 - (b) Consider a partition Π : $a = t_0 < t_1 < \ldots < t_m = b$ of the interval [a, b]. In the context of geometric Brownian motion, increments of Y_t are the ratios $\frac{Y_{t_i+1}}{Y_{t_i}}$ and $\frac{Y_{t_j+1}}{Y_{t_j}}$. They summarize the percentage of, rather than absolute, change. Show that they are independent if $i \neq j$.
 - (c) Do you expect $Y_{t_{i+1}} Y_{t_i}$ and $Y_{t_{j+1}} Y_{t_j}$ to be independent? An intuitive agrument is sufficient.
 - (d) Find the expected value of the increments of Y_t.Hint: You may use the following result about the expected value of the exponential of a normal random variable:

Fact. Suppose that f is a normal variable with expected value μ and variance σ^2 . Then

$$E(e^f) = e^{\mu + \frac{1}{2}\sigma^2}.$$