# Bowdoin College

Math 3603: Advanced Analysis
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## Homework 10

1. In class, we defined the first variation of a function over an interval $[a, b]$ as

$$FV(f)[a,b] = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|,$$

where $\Pi$ is a partition of $[a, b]$ with breakpoints $\{t_i\}$. It computes the distance travelled by a particle whose position at time $t$ is given by $f(t)$. Show that if $f$ is a function which is differentiable everywhere and whose derivative is continuous, then

$$FV(f)[a,b] = \int_a^b |f'(t)| \, dt.$$

**Hint:** Use the Mean Value Theorem.

2. We have the following theorem about functions of finite first variation from class:

**Theorem.** *Suppose $f : [a, b] \to \mathbb{R}$ has $FV(f)[a, b] < \infty$. Then there exist monotone increasing functions $g$ and $h$ such that*

$$f = g - h.$$

Our proof was constructive, meaning we actually found functions $g$ and $h$ that work. However, we had a lot of choice as this decomposition of $f$ is not unique. If $c$ is a constant, both $g + c$ and $h + c$ are monotone increasing and their difference is still equal to $f$. The following question suggests itself: Is the decomposition unique up to this constant? Prove it if so, and provide a counterexample otherwise.

3. Suppose that $B_t$ is standard Brownian motion and define $X_t = X_0 + \mu t + \sigma B_t$ for constants $X_0$, $\mu$, and $\sigma$. Further, let $Y_t = e^{X_t}$. The stochastic process $X_t$ is called $(\mu, \sigma^2)$-Brownian motion and $Y_t$ is the corresponding *geometric Brownian motion*.

   (a) Find the expected value and variance of the increments of $X_t$. Are they independent?

   (b) Consider a partition $\Pi : a = t_0 < t_1 < \ldots < t_m = b$ of the interval $[a, b]$. Explain why the ratios $\frac{Y_{t_{i+1}}}{Y_{t_i}}$ and $\frac{Y_{t_{j+1}}}{Y_{t_j}}$ are independent if $i \neq j$.

   (c) Find the expected value of the increments of $Y_t$.
   **Hint:** You may use the following result about the expected value of the exponential of a normal random variable:

   **Fact.** *Suppose that $f$ is a normal variable with expected value $\mu$ and variance $\sigma^2$. Then*

$$E(e^f) = e^{\mu + \frac{1}{2}\sigma^2}.$$