Bowdoin College

Math 3603: Advanced Analysis
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Homework 1

1. Assume the existence of a function μ from the bounded subsets of $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$ which satisfies

$$\mu([a,b]) = b - a \text{ and } \mu(A \sqcup B) = \mu(A) + \mu(B).$$

We have shown that every countable subset S of $\mathbb{R}$ must have $\mu(S) = 0$, that is, it has *measure zero.* Use the Cantor set C to show that even an uncountable set can have measure zero. That is, show

(a) $\mu(C) = 0$, and

(b) C is uncountable.

2. Let β^* be the set of all infinite sequences of Hs and Ts, that is, the set of *Bernoulli sequences*, and let

$$\beta = \{s \in \beta^* \mid s \text{ does not degenerate to all } T\text{s}\}.$$

Show that $\beta^* \setminus \beta$ is countable. From a measure-theoretic perspective, this allows us to use β instead of β^* when studying the behavior of Bernoulli sequences.

3. Let $R_k : I \to \mathbb{R}$ be the kth Rademacher function and define a product

$$f \cdot g = \int_0^1 fg$$

for Riemann-integrable functions f, g. Show that $R_k \cdot R_k = 1$ for all natural numbers k, and furthermore, $R_k \cdot R_l = 0$ whenever $k \neq l$. If $\cdot$ is viewed as an inner product on the set of Riemann-integrable functions, this exercise shows that the set of Rademacher functions is an orthonormal set.

4. Prove the following lemma.

Lemma. *Let I be the unit interval in $\mathbb{R}$. If f is a non-negative step function on I and $\alpha > 0$, show that*

$$\mu(\{a \in I \mid f(a) > \alpha\}) \leq \frac{1}{\alpha} \int_0^1 f.$$

5. Consider some infinite set X and let $\mathcal{R}$ be the following collection of sets:

$$A \in \mathcal{R} \text{ iff } A \text{ is finite or } A^c \text{ is finite.}$$

Let μ be a set function on $\mathcal{R}$ defined by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite. Examine the following to decide whether μ is a measure.

(a) If X is countable, is μ additive? countably additive?

(b) If X is uncountable, is μ additive? countably additive?