Bowdoin College

Math 2603: Introduction to Analysis
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Homework 12 Solutions

1. The goal of this sequence of exercises is to derive the power rule for differentiation. It begins with a formal definition of the natural logarithm.

   (a) Define a differentiable function $L : \mathbb{R}_{>0} \to \mathbb{R}$ by requiring that

   i. $L'(x) = \frac{1}{x}$, and
   ii. $L(1) = 0$.

   Prove that these conditions define L uniquely. That is, if M is another function satisfying both of the above, then $L(x) = M(x)$ for all $x \in \mathbb{R}_{>0}$. We will write $\ln(x)$ instead of $L(x)$.

   Solution: Assume that $M(x)$ is another function with $M'(x) = \frac{1}{x}$ and $M(1) = 0$. Consider the function $f = L - M$. It is also differentiable and the its derivative is the difference of the derivatives of L and M. The properties of L and M imply $f'(x) = 0$ for all $x \in \mathbb{R}_{>0}$ and $f(1) = 0$. The first statement implies that f is constant, and the second implies this constant is zero, i.e. $f(x) = L(x) - M(x) = 0$ for all $x \in \mathbb{R}_{>0}$, as desired.

   (b) Show that $\ln x$ is a bijection from $\mathbb{R}_{>0}$ to $\mathbb{R}$.

   Solution: We first show that $\ln x$ is injective. Suppose not, and assume $\ln x = \ln y$ for some numbers x and y. Since $[x, y]$ is compact and $\ln x$ is continuous (it is, after all, differentiable), it attains a maximum and a minimum at some points, say a and b. If, as sets, $\{a, b\} = \{\ln x, \ln y\}$, then $\ln x$ must be constant, which cannot be true since it's derivative is not equal to zero anywhere.

   Otherwise, one of the points a or b must lie in the interior of the interval $[\ln x, \ln y]$, and by a theorem from class, the derivative of $\ln x$ must equal zero there. This is again a contradiction, since the derivative of $\ln x$ is always positive.

   To show that $\ln x$ is surjective, we use the fact that its image has no upper or lower bound. Knowing this, let $M \in \mathbb{R}$. Then there is a point x_1 for which $f(x_1) > M$ and another point at which $f(x_2) < M$ (otherwise, the image of f would be bounded, either above or below. Since $\ln x$ is continuous, the intermediate value theorem implies there is a point c between x_1 and x_2 for which $f(c) = M$. Since this can be done for every $M \in \mathbb{R}$, $\ln x$ is surjective.

   To finish, we must show that $\ln x$ is not bounded above or below. We focus on "above." We know that $\ln 1 = 0$ with slope at least $\frac{1}{2}$ on the interval $[1, 2]$. This means $\ln 2 \geq \frac{1}{2}$. [1] Similarly, its slope is at least $\frac{1}{3}$ on $[2, 3]$ so that $\ln 3 \geq \frac{1}{2} + \frac{1}{3}$. Continuing in this manner,

   $$\ln n \geq \sum_{2}^{n} \frac{1}{n}.$$

   Since the latter diverges to infinity, $\ln x$ has no upper bound.

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[1] If you are a stickler, this fact requires proof. If $\ln 2 < \frac{1}{2}$, the Mean Value Theorem would guarantee a point $c \in (1, 2)$ where $\ln'(c) = \frac{\ln(2) - \ln(1)}{2 - 1} = \ln(2) < \frac{1}{2}$. But we know that for all $c \in (1, 2)$, $\ln'(c) = \frac{1}{c} > \frac{1}{2}$. Hence $\ln 2 \geq \frac{1}{2}$.

(c) Since $\ln x$ is a bijection, it has an inverse function, which we define to be e^x. Using the chain rule, prove that $(e^x)' = e^x$.

Solution: Note that $e^{\ln(x)} = x$. Using the chain rule:

$$(e^{\ln x})' = e'(\ln(x)) \cdot \frac{1}{x} = 1$$

Let $y = \ln x$, so that $x = e^y$. Our equation can be rewritten as

$$(e^y)' \cdot \frac{1}{e^y} = 1$$

or, in other words, $(e^y)' = e^y$, as desired.

(d) For a positive real number x and any real α, we can now define x^α as

$$x^\alpha = e^{\alpha \ln x}.$$

Armed with this definition, show that

$$(x^\alpha)' = \alpha x^{\alpha - 1}.$$

Solution: Note that

$$(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \cdot \alpha \frac{1}{x} = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha x^{\alpha-1}.$$

2. Suppose that f is differentiable at every point of $[a, b]$ and suppose that the derivative is never zero. Prove that f is strictly monotonic on $[a, b]$. Note that f' is *not* assumed to be continuous.

Solution: Suppose that f is not strictly monotonic. Without loss of generality, there are points x_1, x_2 and x_3 in the interval for which

$$f(x_1) \leq f(x_2) \geq f(x_3).$$

Again, without loss of generality, assume that $f(x_1) \leq f(x_3)$. Since f is differentiable and therefore continuous, by the intermediate value theorem there is a point $x_4 \in [x_1, x_2]$ where $f(x_4) = f(x_3)$. By the mean value theorem for derivatives, there most be a point $c \in [x_4, x_3]$ where $f'(c) = 0$. This is a contradiction, so f must be strictly monotonic.

3. From class, we know that bounded continuous functions on a compact interval in $\mathbb{R}$ are Riemann integrable. The following exercise will show that a function can have one jump or removable discontinuity and still remain Riemann integrable. Consider an interval $[a, b] \in \mathbb{R}$ and a point $c \in [a, b]$. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = 0$ unless $x = c$, when $f(c) = 1$. In other words, $f = \chi_{\{c\}}$, the indicator function of the set $\{c\}$.

(a) Show that f is Riemann integrable on $[a, b]$.

Solution: Consider a partition P of the interval $[a, b]$. The lower sum $L(f, P)$ is zero, and if c lies in the ith subinterval of P, the upper sum $U(f, P)$ equals Δx_i. Given $\epsilon > 0$, let P be a partition for which $\Delta x_i < \epsilon/2$. Then $U(f, P) = \epsilon/2$. But this means

$$U(f, P) - L(f, P) = \epsilon/2 - 0 = \epsilon/2 < \epsilon.$$

By a theorem from class, this shows that f is Riemann integrable.

Alternately, one can show that $U(f) = L(f) = 0$ by a similar argument.

(b) Conclude that any function which is continuous except for possibly for a jump or a removable discontinuity at one point is Riemann integrable.

Solution: If g has a removable discontinuity at $c \in [a, b]$, then g can be written as $g = \lambda f + h$ where $\lambda \in \mathbb{R}$, f is the function from the first part of the problem, and h is continuous. Since both λf and h are Riemann integrable, so is g.

If, on the other hand, g has a jump discontinuity at $c \in [a, b]$, then g is Riemann integrable on $[a, c]$ and $[c, b]$; it is continuous on one of these intervals, and has a removable discontinuity on the other. By a theorem from class, g must be Riemann integrable on all of $[a, b]$.

In fact, by induction one can extend this exercise to show that a finite number of such discontinuities do not affect Riemann integrability. Consequently, things like step functions are Riemann integrable as well. Can this requirement be relaxed even further?

The answer is "yes", and in fact, by quite a bit. The complete answer was found by Henri Lebesgue in his doctoral thesis.[2] The complete answer involves the measure of a set. We will say a subset of $\mathbb{R}$ has *measure zero* if for every $\epsilon > 0$, it can be covered by a countable number of open intervals whose total length is less than ϵ. It turns out that the rational numbers and the Cantor set both have measure zero. Here is Lebesgue's observation:

Theorem (Riemann-Lebesgue Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and let D be the set of points where it is discontinuous. Then $f \in \mathcal{R}[a, b]$ if and only if D has measure zero.*

We will not have a chance to prove this in this class, so you will have to refrain using this result in what follows!

4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that $\int_a^b f = 0$ iff f is the zero function.

Solution: The reverse direction is easy, so let's prove the forward direction. Assume that $\int_a^b f = 0$ but that $f \neq 0$; that is, for some $c \in [a, b]$, $f(c) > 0$. Let $\epsilon = f(c)/2$. Since f is continuous, there is a $\delta > 0$ so that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$, or equivalently, $|f(x)| \geq f(c)/2$.

Let P be a partition that includes the interval $[c - \delta, c + \delta]$. Then $L(f, P) > f(c)/2 \cdot \delta > 0$. But since $L(f) \geq L(f, P)$, this is a contradiction and f must equal zero everywhere.

[2] Therein, he also constructed what is known today as the *Lebesgue integral* that has come to completely supercede the Riemann integral. But this is a story for Math 3603.