Bowdoin College

Math 2603: Introduction to Analysis
Prof. Thomas Pietraho
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Homework 9

1. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that if f is continuous, then in fact $f(x) = 0$ for all $x \in \mathbb{R}$!

2. This problem revisits the notion of an additive homomorphism, this time with a slightly enlarged domain. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive homomorphism; that is, it satisfies

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

   Describe the set of all possibilities for f if we assume that it is continuous.

   Extra Credit: What are the possibilities for f if we don't assume that it is continuous?

3. Let (S, ρ) be a metric space endowed with the discrete metric. Describe all of its connected subsets.

4. Suppose that $f : S_1 \to S_2$ is a continuous bijection between metric spaces (S_1, ρ_1) and (S_2, ρ_2). Since f is a bijection, it has an inverse.

   Question: Is f^{-1} is continuous as well?

   There are certain important settings when the answer is "yes." For instance, if $(S_1, \rho_1) = (S_2, \rho_2) = (\mathbb{R}, |\cdot|)$ this result is called the Invariance of Domain Theorem. Unfortunately, there are settings where the answer may be "no." Find metric spaces S_1 and S_2 and a continuous bijection between them whose inverse is not continuous.

   Hint: One possible example uses a discrete metric space.