Bowdoin College

Math 2603: Introduction to Analysis
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Homework 8

1. The original definition of a continuous function is fairly cumbersome for daily use. The goal of this exercise is to verify a somewhat more palatable characterization of continuous functions. Let (S_1, ρ_1) and (S_2, ρ_2) be metric spaces and consider $f : S_1 \to S_2$. Show that f is continuous iff for every open set $U \subset S_2$, the inverse image $f^{-1}(U)$ in S_1 is also an open set.

2. Consider a function $f : S \to \mathbb{R}$ from some metric space (S, ρ) into the real numbers. Define another function $|f| : S \to \mathbb{R}$ by letting

$$|f|(x) = |f(x)| \qquad \forall x \in S.$$

   (a) Suppose that f is continuous; does that imply that $|f|$ is continuous as well?
   Hint: from class we know that continuity is preserved under composition of functions.

   (b) Suppose that $|f|$ is continuous; does that imply that f is continuous as well?

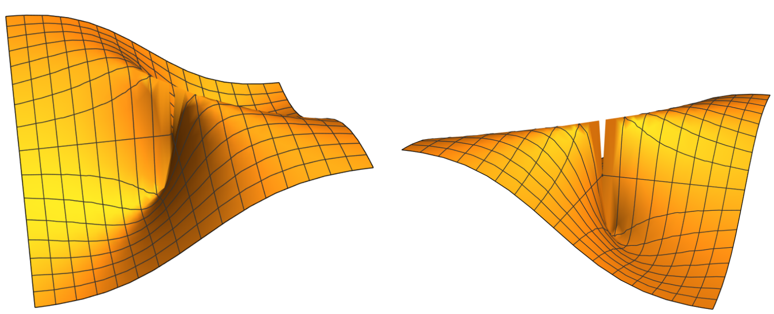
3. Consider a function $f : S_1 \to S_2$. Show that if x is not a limit point of S_1, then f must be continuous at x.

4. Show that every polynomial $p : \mathbb{R} \to \mathbb{R}$ is a continuous function.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   The plot of this function is somewhat interesting. Below are two perspectives:



   Determine whether $f(x,y)$ is continuous at the origin $(0,0)$ and justify your answer.