Bowdoin College

Math 2603: Introduction to Analysis
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Homework 7

1. Recall the following definition from class:

   Definition. *Suppose that* (S, ρ) *is a metric space. A function* $f : S \to S$ *is a contraction map if there is a constant* $0 < \alpha < 1$ *such that*

   $$\rho(f(x), f(y)) \leq \alpha\rho(x, y).$$

   Intuitively, a contraction map shrinks the distance between points of our metric space by at least a factor of α. One of the more important theorems we prove in the course shows that for every contraction map f on a complete metric space, there is a point $x \in S$ such that $f(x) = x$. However, if we relax the restriction on f and require only that

   $$\rho(f(x), f(y)) < \rho(x, y)$$

   for all distinct $x, y \in S$, then a fixed point may not exist. Find an example of such a function f and make sure to verify your assertions.

2. Let A be an $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$. Define a function $T : \mathbb{R}^n \to \mathbb{R}^n$ by

   $$T(\vec{x}) = A\vec{x} + \vec{b}.$$

   We can think of points in $\mathbb{R}^n$ as *states* in a physical system, each entry representing one of its parameters. For example, $\vec{x}$ can represent the present state of the economy and each entry one of the leading economic indicators. One can model how this system evolves over time by forming a sequence $\vec{x_0} = \vec{x}$ and defining the successive states recursively by

   $$\vec{x}_{i+1} = T(\vec{x}_i).$$

   A recurrent question in a number of disciplines is the following:

   > **Question:** Does the physical system described by the iterations of the transformation T reach an equilibrium? That is, is there a vector $\vec{x}$ so that $T(\vec{x}) = \vec{x}$, or in other words, does T have a fixed point?

   The Contraction Mapping Theorem hints at one possible answer. The object of this exercise is to find out under what circumstances the transformation T is a contraction map.

   (a) Consider the metric space $(\mathbb{R}^n, \rho_\infty)$. Write $A = (a_{ij})$ for a matrix in $\mathbb{R}^n$ and let $\vec{b} = (b_1, b_2, \ldots, b_n)$. Show that the map T defined above is a contraction map if there exists an $\alpha \in [0, 1)$ such that

   $$\sum_j |a_{ij}| \leq \alpha$$

   for every value of i. In other words, T is a contraction map if for every row of A, the sum of the absolute values of its entries is less than one.

   Hint: Let $\vec{y} = T(\vec{x})$ and write the coordinates of $\vec{y}$ in terms of the a_{ij}, x_i, and b_i.

(b) Now consider the metric space $(\mathbb{R}^n, \rho_2)$. Again write $A = (a_{ij})$ for a matrix in $\mathbb{R}^n$ and let $\vec{b} = (b_1, b_2, \ldots, b_n)$. Show that the map T defined above is a contraction map if there exists an $\alpha \in [0, 1)$ such that

$$\sum_{i,j} a_{ij}^2 \leq \alpha.$$

That is, T is a contraction map if the sum of squares of the entries of A is less than one.

Hint: Use the Cauchy-Schwarz inequality.

(c) Find a matrix A that satisfies the condition derived in (a) but not the one derived in (b); and vice-versa.

(d) The restrictions derived in (a) and (b) are satisfied by two different sets of matrices. Nevertheless, the map T associated with a matrix A in *either* set will have a fixed point! How could you use the ideas in (a) and (b) to expand the set of transformations T that have a fixed point even further?

3. Suppose that (S, ρ) is a metric space and let $f : S \to S$ be the identity function defined by $f(s) = s$ for all $s \in S$. Show that f is continuous by using the our original definition of continuity.

4. Suppose that (S, ρ) is a metric space and let $f : S \to S$ be a contraction mapping, so that there is a positive constant $\alpha < 1$ such that for all distinct $x, y \in S$, we have

$$\rho(f(x), f(y)) \leq \alpha \rho(x, y).$$

Show that f is a continuous function.

5. Show that the absolute value function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous.