Bowdoin College

Math 2603: Introduction to Analysis
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Fall, 2022

Homework 6

1. Consider a metric space S with metric ρ. If A and B are subsets of S, we can define the distance between them to be
$$\rho(A, B) = \inf_{a \in A, b \in B} \rho(a, b);$$
that is, the infimum of the set of all possible distances between a point in A and a point in B. Show that $\rho(A, B) = 0$ if $A \cap B \neq \varnothing$ and, by providing a counterexample, show that the converse need not hold.

2. Suppose that (S, ρ) is a metric space. If T is a subset of S and t is a limit point of the set T, show that there is a sequence of points $\{p_n\}_{n=1}^{\infty}$ in T such that
$$\lim p_n = t.$$

3. Consider a sequence $\{s_n\}$ of real numbers satisfying $a \leq s_n \leq b$ for some $a, b \in \mathbb{R}$ and all $n \in \mathbb{N}$. Suppose that the sequence converges and that $\lim s_n = s$. Show that $a \leq s \leq b$.

4. We will say that a sequence of real numbers is *bounded* iff its elements form a bounded subset of $\mathbb{R}$.

   (a) Show that if $\{c_n\}$ is a bounded sequence of real numbers, then there is a positive number $D \in \mathbb{R}$ such that $|c_n| \leq D$ for all $n \in \mathbb{N}$.

   (b) Let $(S, \rho) = (\mathbb{R}, |\cdot|)$ and consider a sequence $\{s_n\}$ such that $\lim s_n = 0$. Suppose further that $\{c_n\}$ is a bounded sequence. Show that the sequence of their products $\{c_n s_n\}$ converges and that
   $$\lim c_n s_n = 0.$$
   Note that we have not assumed that $\{c_n\}$ converges!

5. In our proof of the convergence of the alternating harmonic series, we will use the fact that for every integer m greater than $n \in \mathbb{N}$, the following inequality holds
$$\Big| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots \pm \frac{1}{m} \Big| < \frac{1}{n}.$$
Prove it.

6. Define a sequence of real numbers $s_n = (1 + \frac{1}{n})^n$. Show that the sequence is monotone and bounded, concluding that it must converge. Its limit is defined as the familiar number:
$$\lim \big(1 + \frac{1}{n}\big)^n = e.$$
Hint: Use the binomial theorem. Also, feel free to use without proof the fact that terms of this sequence are always smaller than 3. But if you can, prove that this bound does indeed hold.