Bowdoin College

Math 2603: Introduction to Analysis
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Homework 5

1. Show that the union of two compact sets is compact.
2. Suppose that T is closed and U is compact. Show that $T \cap U$ is compact.
3. Show that if T_i is compact for every $i \in I$, then so is the intersection $\bigcap_{i \in I} T_i$.
4. The Heine-Borel Theorem shows that all closed and bounded subsets of the metric space $\mathbb{R}$ are compact. Its proof uses the least upper bound property of $\mathbb{R}$ in an essential way. Hence it is a natural question to examine whether such a theorem is still true in a metric space which does not satisfy the least upper bound property. This is the aim of the final question:

   Question: Consider the metric space $\mathbb{Q}$ with the usual metric. Let $T = (a, b) \cap \mathbb{Q}$ where both a and b are irrational numbers. Show that T is both closed and bounded, but that it is not compact.

5. Prove the following proposition. We will use it a number of times in the course to show that two points of a metric space are equal.

   > **Proposition.** *Consider points p and p' in a metric space (S, ρ). If $\rho(p, p') < \epsilon$ for every $\epsilon > 0$, then $p = p'$.*