Bowdoin College

Math 2603: Introduction to Analysis
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Fall, 2022

Homework 4

1. Prove the following proposition:

   Proposition: Let (S, ρ) be a metric space. Then

   (a) S is a closed set,

   (b) $\varnothing$ is a closed set,

   (c) If I is a finite set and A_i is a closed set for all $i \in I$, then $\bigcup_{i \in I} A_i$ is a closed set,

   (d) If A_i is a closed set for all $i \in I$, then $\bigcap_{i \in I} A_i$ is a closed set.

   Hint: The key here lies in DeMorgan's Laws. If you have not seen them yet in a prior class like Math 2020, feel free to look them up. Just make sure to read a mathematical version; there is a variant that deals with propositional logic that is harder to digest. Their proof is not necessary, you should treat them as known.

2. Consider a metric space (S, ρ) and a point $p \in S$. Prove that $\{p\}$ is a closed set in S. Conclude that in every metric space, a set of the form $\{p_1, p_2, \ldots, p_n\}$ is closed.

3. Write $\overline{T}$ for the closure of a set in an arbitrary metric space (S, ρ). Show that $\overline{T}$ is the smallest closed set containing T.

4. For any subset $T \subset \mathbb{R}$, recall the notions of the complement $T^c = \mathbb{R} \setminus T$ and closure $\overline{T}$. One can form new sets from T by taking its complement, its closure, and then the complements and closures of the new sets, and so on. What is the largest number of distinct sets which can be obtained from a subset of $\mathbb{R}$ by taking successive complements and closures in this way? Be creative in the sets you construct.

   > **Example:** For instance, if we start with the set $T = [0, \infty)$, taking successive complements and closures we obtain the sets $[0, \infty)$, $(-\infty, 0)$, $(-\infty, 0]$, and $(0, \infty)$, for a grand total of four distinct sets.

   For your answer, simply write down the set T you have found and the family of distinct sets that are generated from T in this way. You will receive full credit if you find T where this family has at least ten members, although the actual maximum is in fact greater than ten.

   Note: If you think you have found the largest number possible and you have some time to spare, you can try to prove your result. Talk to me if you would like a hint, it is not impossibly hard, but you have to think of the problem just the right way.