BOWDOIN COLLEGE

Math 3603: Advanced analysis Prof. Thomas Pietraho

Homework 4

1. This exercise shows that Lebesgue measure on \mathbb{R} is *translation invariant*. Consider $c \in \mathbb{R}$. Given any subset A of \mathbb{R} , define

 $A + c = \{ x \in \mathbb{R} \mid x - c \in A \}.$

This is just a translation of the set A by c. Prove that if A is Lebesgue-measurable, then A + c is also Lebesgue-measurable and

$$\mu_L(A+c) = \mu_L(A).$$

Hint: First show this is true for intervals and then verify that the equation holds for outer measure.

2. Let X be a set, \mathcal{F} a σ -field of subsets of X, and μ a probability measure on \mathcal{F} . Consider a sequence of sets $\{A_i\} \in \mathcal{F}$. The purpose of this exercise is to detail the relationship between the sets $\{A_i, i.o.\}$ and $\{A_i, a.a.\}$. To that effect, show that

$$\liminf A_i^c = (\limsup A_i)^c.$$

Conclude that $\mu(\liminf A_i^c) = 0$ iff $\mu(\limsup A_i) = 1$.

Definition: Recall the map we established between Bernoulli sequences β and points in the unit interval *I*. If μ_L is Lebesgue measure and \mathcal{M} is the family of all Lebesguemeasurable subsets of *I*, then (I, \mathcal{M}, μ_L) is a probability space. We will say that an event *E* occurring on Bernoulli sequences is **plausible** if the corresponding subset $B_E \in \mathcal{M}$.

3. Show that a gambler quadrupling his initial stake is a plausible event in the perpetual cointossing game where he wins one dollar if the coin flip is an H and loses a dollar if the coin flip is a T.

Note: A gambler cannot quadruple his initial stake if he loses all his money beforehand.

4. Let N be a non-zero integer. Prove that a random walk on the line starting at zero passes either through the point N or the point -N with probability one. Conclude that it passes through N with probability at least one-half.