Bowdoin College

Math 2603: Introduction to Analysis
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Homework 3

1. Consider vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$. We can define a function $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by letting

$$\rho_1(\vec{v}, \vec{w}) = \sum_{i=1}^{n} |v_i - w_i|.$$

In $\mathbb{R}^2$ this is called the "taxicab metric." Prove that when endowed with this metric, $\mathbb{R}^2$ becomes a metric space.

2. Recall the definitions of the metrics ρ_2, ρ_1, and ρ_∞ on the set $\mathbb{R}^n$ from class. They are a part of a larger family of metrics called the p-metrics.

Definition. *Let $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$. For $p \in \mathbb{N}$, we define a function*

$$\rho_p(\vec{v}, \vec{w}) = \sqrt[p]{|v_1 - w_1|^p + \ldots |v_n - w_n|^p}.$$

It turns out that this is always a metric. I urge you to stay away from the proof of the triangle inequality.[1]. Armed with this knowledge, try do visualize how these metrics are interrelated by the following exercise. Let $\vec{0}$ be the origin in $\mathbb{R}^n$. Draw the unit sphere $S_1(\vec{0})$ in the metric spaces $(\mathbb{R}^2, \rho_2)$, $(\mathbb{R}^2, \rho_1)$, $(\mathbb{R}^2, \rho_\infty)$, $(\mathbb{R}^3, \rho_1)$, and $(\mathbb{R}^2, \rho_3)$.

3. Recall the field $\mathbb{Q}[\sqrt{2}]$ from the previous assignment. Since all of its elements are real numbers, it inherits a the usual ordering from $\mathbb{R}$ and it is easy to verify that $\mathbb{Q}[\sqrt{2}]$ is also an ordered field. Find another way of ordering the elements of $\mathbb{Q}[\sqrt{2}]$, with the new order denoted by $\prec$, so that $(\mathbb{Q}[\sqrt{2}], \prec)$ is still an ordered field.

4. Show that it is possible to find a metric space (S, ρ), points $x, y \in S$, and positive numbers r_1 and r_2 so that

$$B_{r_1}(x) \subsetneq B_{r_2}(y),$$

and $r_1 > r_2$; that is, a ball with a smaller radius contains a ball with a larger radius (!).

5. The following result will be crucial when we start looking at point-set topology. Recall that if $A \subset X$ and $C \subset Y$, then

$$f(A) = \{y \in Y \mid f(x) = y \text{ for some } x \in A\},$$

and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

Note that f need not be invertible for $f^{-1}(C)$ to be defined.

Theorem: Suppose that $f : X \to Y$ is a function. Then
(a) If $A, B \subset X$, then $f(A \cup B) = f(A) \cup f(B)$.
(b) If $A, B \subset X$, then $f(A \cap B) \subset f(A) \cap f(B)$.

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[1] You've been warned

(c) If $A, B \subset X$, then $f(A \setminus B) \supset f(A) \setminus f(B)$.
(d) If $C, D \subset Y$, then $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(e) If $C, D \subset Y$, then $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
(f) If $C, D \subset Y$, then $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$.

So set operations "play nicely" with images and inverse images, *except* for two cases. Prove that the second and third parts of the theorem are true, and find a counterexample in each of these two cases that shows we don't always have equality in those two cases.