BOWDOIN COLLEGE

Math 3603: Advanced analysis Prof. Thomas Pietraho

Homework 3

1. Let X be a set, \mathcal{R} be a ring of subsets of X and μ a measure on \mathcal{R} . Let μ^* be the corresponding outer measure. It is possible but a bit tedious to prove the following:

Theorem: If $A \in \mathcal{R}$, then for every $E \subset X$, $\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E).$

Taking this theorem as given, prove the following corollary of this result:

Corollary: If $A \in \overline{\mathcal{R}}$, then for every $E \subset X$, $\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E).$

- 2. Consider a set X together with a ring \mathcal{R} of subsets endowed with measure μ . As in class, construct the family of measurable subsets \mathcal{M} of X with measure μ^* . Show that \mathcal{M} is a σ -ring.
- 3. Recall that $A \in \overline{\mathcal{R}}$ iff $A = \lim_{n \to \infty} A_n$ with each $A_n \in \mathcal{R}$. Show that
 - (a) the Cantor set, and
 - (b) the rational numbers,

lie in $\overline{\mathcal{R}}$ when $X = \mathbb{R}$ and $\mathcal{R} = \mathcal{R}_{Leb}$ by finding a sequence of finite unions of intervals that converges to each one.

4. Recall that a set T is dense in a metric space X if there is a sequence of elements of T converging to every element of X. The purpose of this exercise is to show that there can a considerable difference between the measures of a set and a dense subset therein.

Consider the metric space $X = \mathbb{R}$, let \mathcal{M} be the collection of Lebesgue-measurable subsets of \mathbb{R} with measure μ_L . Show that for any $\delta > 0$ there is an open dense subset U of \mathbb{R} with $\mu_L(U) < \delta$.