Bowdoin College

Math 2603: Introduction to Analysis
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Homework 2

1. Define the set of real numbers

   $$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$

   Show that together with the usual addition and multiplication, $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ forms a field. You may freely use that fact that $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{R}, +, \cdot)$ are fields.

2. Suppose that $(F, +, \cdot)$ is a field and x, y, and z are elements of F. Show that

   (a) If $x \neq 0$ and $xy = xz$, then $y = z$.

   (b) If $x \neq 0$ and $xy = x$, then $y = 1$.

3. Suppose that $(F, +, \cdot)$ is an ordered field with $x, y \in F$. Show that

   (a) If $x < y$ then $-y < -x$.

   (b) $1 > 0$.

4. Can the set of complex numbers $\mathbb{C}$ form and ordered field?
   Hint: Consider the relationship between i and 0.

5. Consider a set $S \subset \mathbb{R}$ and suppose that it has a lower bound. Define the set $-S$ as $\{-r \mid r \in S\}$.

   (a) Define precisely the notion of a greatest lower bound for a set S and explain why one exists if S has a lower bound. The greatest lower bound for a set S is often written as $\inf S$, the infimum of S.

   (b) Show that $\inf S = -\sup(-S)$.

6. A map $f : \mathbb{Q} \to \mathbb{Q}$ is called an *additive homomorphism* if it satisfies

   $$f(a + b) = f(a) + f(b).$$

   What are all the additive homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$?

7. Does there exist a finite ordered field?