Bowdoin College

Math 2603: Introduction to Analysis
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Homework 13

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \in (0,1] \text{, and} \\ 0 & \text{if } x = 0 \end{cases}$$

It turns out that f is not continuous at zero, but the discontinuity is neither a jump or removable[1]. Nevertheless, prove that $f \in \mathcal{R}[0,1]$.

2. Suppose that f is a continuous real-valued function on the interval $[a,b]$. Prove that these exists a point $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

This is the *Mean Value Theorem for Integrals.*

3. Recall the following definition from class:

Definition. *A sequence of functions $\{f_n\}$ from a set S to the real numbers is said to converge uniformly to a function f iff for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies that*

$$\sup_{x \in S} |f(x) - f_n(x)| < \epsilon.$$

This is not a completely new notion. If the functions are also assumed to be continuous and S is a compact metric space, then a sequence of functions $\{f_n\}$ converges to f uniformly if and only if it converges to f under the sup metric. A recurring theme in this week's lectures was that a lot of reasonable properties of the functions in the sequence are passed to the limit function, but one has to be careful. The following exercises investigate the "limits" of this philosophy.

(a) Suppose that the sequence of functions $\{f_n\}$ converges uniformly to the function f and the sequence of functions $\{g_n\}$ converges uniformly to g.

i. Show that $\{f_n + g_n\}$ converges uniformly to $f + g$.

ii. We begin with a definition.

Definition. *A sequence of functions $\{f_n\}$ is bounded if there is a number B such that $|f_n(x)| < B$ for all x and all $n \in \mathbb{N}$.*

Show that if $\{f_n\}$ and $\{g_n\}$ are bounded, then $\{f_n g_n\}$ converges uniformly to fg.

iii. Show that the boundedness restriction in the above statement is necessary. That is, find examples of f_n and g_n that converge uniformly to functions f and g, but whose product does not converge uniformly to fg.

(b) **Extra Credit:** Prove the Cauchy Criterion for Uniform Convergence:

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[1] Points where a bounded function is discontinuous but the discontinuity is neither a jump or removable are called *essential discontinuities.*

Theorem. *The sequence* $\{f_n\}$ *converges uniformly on* S *iff* $\forall \epsilon > 0$, $\exists N$ *such that* $n, m > N$ *and* $x \in S$ *implies*

$$|f_n(x) - f_m(x)| < \epsilon.$$

This is a crucial lemma which will be necessary to show that the space of continuous bounded functions on a metric space is a complete metric space when endowed with the uniform metric.

4. The following exercise forms the backbone for the so-called *method of moments* used in probability and statistics, although it has a variety of other applications. If f is a function defined on $[0, 1]$, its nth moment M_n is defined as

$$M_n = \int_0^1 f(x)x^n \, dx = 0.$$

Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that $M_n = 0$ for every non-negative integer n. Prove that f must equal to zero on $[0, 1]$.

Hint: The hypothesis implies that the integral of the product of f with any polynomial is zero. Use the Weierstrass Approximation Theorem to show that $\int_0^1 f^2(x) \, dx = 0$ from which you should be able to deduce that $f(x) = 0$.

Stray Observation: For those of you familiar with a bit of fancy linear algebra, this is morally very similar to showing that if the inner product of a vector with every element of a basis is zero, then the vector itself must be zero. But don't use linear algebra for this problem; there are too many tricky details.