Bowdoin College

Math 2603: Introduction to Analysis
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Homework 12

1. The goal of this sequence of exercises is to derive the power rule for differentiation. It begins with a formal definition of the natural logarithm.

   (a) Define a differentiable function $L : \mathbb{R}_{>0} \to \mathbb{R}$ by requiring that

      i. $L'(x) = \frac{1}{x}$, and

      ii. $L(1) = 0$.

   Prove that these conditions define L uniquely. That is, if M is another function satisfying both of the above, then $L(x) = M(x)$ for all $x \in \mathbb{R}_{>0}$. We will write $\ln(x)$ instead of $L(x)$.

   (b) Show that $\ln x$ is a bijection from $\mathbb{R}_{>0}$ to $\mathbb{R}$.

   (c) Since $\ln x$ is a bijection, it has an inverse function, which we define to be e^x. Using the chain rule, prove that $(e^x)' = e^x$.

   (d) For a positive real number x and any real α, we can now define x^α as

   $$x^\alpha = e^{\alpha \ln x}.$$

   Armed with this definition, show that

   $$(x^\alpha)' = \alpha x^{\alpha - 1}.$$

2. Suppose that f is differentiable at every point of $[a, b]$ and suppose that the derivative is never zero. Prove that f is strictly monotonic on $[a, b]$. Note that f' is *not* assumed to be continuous.

3. From class, we know that bounded continuous functions on a compact interval in $\mathbb{R}$ are Riemann integrable. The following exercise will show that a function can have one jump or removable discontinuity and still remain Riemann integrable. Consider an interval $[a, b] \in \mathbb{R}$ and a point $c \in [a, b]$. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = 0$ unless $x = c$, when $f(c) = 1$. In other words, $f = \chi_{\{c\}}$, the indicator function of the set $\{c\}$.

   (a) Show that f is Riemann integrable on $[a, b]$.

   (b) Conclude that any function which is continuous except for possibly for a jump or a removable discontinuity at one point is Riemann integrable.

   In fact, by induction one can extend this exercise to show that a finite number of such discontinuities do not affect Riemann integrability. Consequently, things like step functions are Riemann integrable as well. Can this requirement be relaxed even further?

   The answer is "yes", and in fact, by quite a bit. The complete answer was found by Henri Lebesgue in his doctoral thesis.[1] The complete answer involves the measure of a set. We will say a subset of $\mathbb{R}$ has *measure zero* if for every $\epsilon > 0$, it can be covered by a countable number of open intervals whose total length is less than ϵ. It turns out that the rational numbers and the Cantor set both have measure zero. Here is Lebesgue's observation:

[1] Therein, he also constructed what is known today as the *Lebesgue integral* that has come to completely supercede the Riemann integral. But this is a story for Math 3603.

Theorem (Riemann-Lebesgue Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and let D be the set of points where it is discontinuous. Then $f \in \mathcal{R}[a, b]$ if and only if D has measure zero.*

We will not have a chance to prove this in this class, so you will have to refrain using this result in what follows!

4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that $\int_a^b f = 0$ iff f is the zero function.