Bowdoin College

Math 3603: Advanced analysis
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Homework 12

1. We defined the Itô stochastic integral as the *mean square limit*, that is the limit in the $L^2(X)$ metric space, of partial sums

$$\int_a^b f(t)\, dB_t = \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i)(B_{t_{i+1}} - B_{t_i})$$

for a partition $\Pi : a = t_0 < t_1 < \dots t_n = b$. This approach uses the left endpoint of each interval. A more general approach would be to choose an intermediate point $\tau_i \in [t_i, t_{i+1}]$ and define a general stochastic integral as

$$\int_a^b f(t)\, dB_t = \lim_{n\to\infty} \sum_{i=0}^{n} f(\tau_i)(B_{t_{i+1}} - B_{t_i})$$

So pick $\alpha \in [0,1]$, let $\tau_i = \alpha t_i + (1-\alpha)t_{i+1}$, and compute

$$E\Big(\int_0^T B_t\, dB_t\Big)$$

using the more general definition. What is the answer for the Itô integral, and how does the answer vary for $\alpha \in [0,1]$? From a finance perspective, why is the value $\alpha = 1$ used for the Itô integral? And again from this perspective, why does your answer make sense when $\alpha < 1$?

Hint: Note that

$$B_{t_{i+1}} - B_{t_i} = B_{t_{i+1}} - B_{\tau_i} + B_{\tau_i} - B_{t_i}$$

and use the formula for covariance of Brownian motion.