Bowdoin College

Math 2603: Introduction to Analysis
Prof. Thomas Pietraho
Fall, 2022

Homework 10

1. Prove that every uniformly continuous function $f : S_1 \to S_2$ is continuous at every point $x_0 \in S_1$.

2. Consider a function $f : S_1 \to S_2$. If f is uniformly continuous and $\{s_n\}$ is a Cauchy sequence of points in S_1, then the sequence $\{f(s_n)\}$ in S_2 is Cauchy as well. Show that this result is not true in general if f is only assumed to be continuous.

3. It is possible to show that the subsets $(0, 1)$ and $[0, 1]$ of $\mathbb{R}$ are equinumerous, that is, there is a bijection between them.

   (a) Does there exist a continuous bijection $f : [0, 1] \to (0, 1)$? Remember that $[0, 1]$ is compact.

   (b) Show that a bijection $f : (0, 1) \to [0, 1]$ cannot be continuous.
   Hint: Use the Intermediate Value Theorem.

4. Consider $\lim_{x\to 0} x \sin(\frac{1}{x})$. It seems very reasonable that this limit should equal zero: x itself approaches zero and $\sin(\frac{1}{x})$ is always between negative one and one, so their product should also approach zero. While this is great intuition, it is not much of a proof. Use the formal ϵ-δ definition to show that indeed
$$\lim_{x\to 0} x \sin(\tfrac{1}{x}) = 0.$$