Introduction to *Introduction to Analysis*

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Main Themes:

Fundamentally, analysis is concerned with two things:

- approximation, and
- convergence.

They are very closely related.

Example 1: Real Numbers

Consider the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

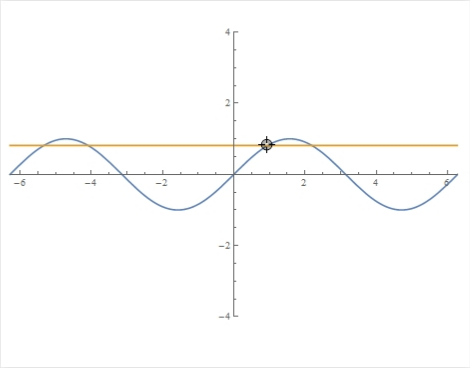
If we graph its points, it is clear visually that the elements approach 1.

But convergence is a phenomenon that can be studied for other mathematical objects.

Example 2: Taylor Series

Consider the sequence of Taylor polynomials centered at $x = 0.5$ approximating the function $f(x) = \sin x$.

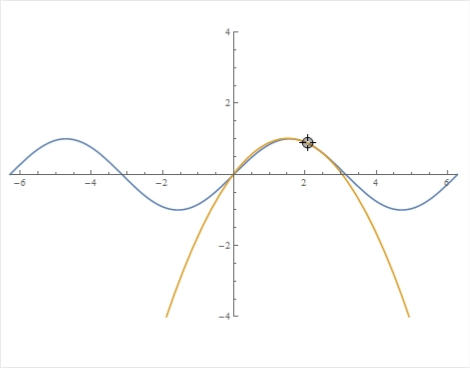
$$n = 0$$



Example 2: Taylor Series

Consider the sequence of Taylor polynomials centered at $x = 0.5$ approximating the function $f(x) = \sin x$.

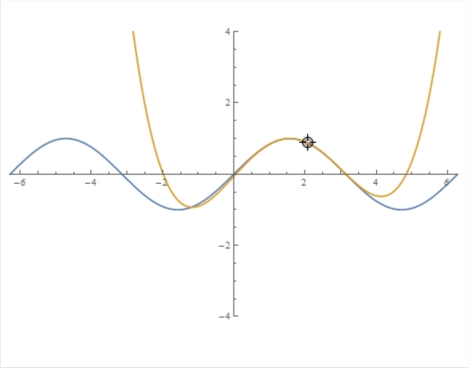
$$n = 2$$



Example 2: Taylor Series

Consider the sequence of Taylor polynomials centered at $x = 0.5$ approximating the function $f(x) = \sin x$.

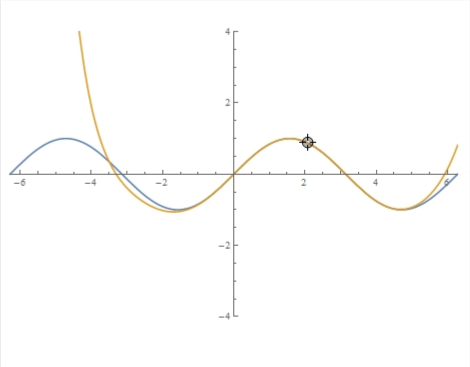
$$n = 6$$



Example 2: Taylor Series

Consider the sequence of Taylor polynomials centered at $x = 0.5$ approximating the function $f(x) = \sin x$.

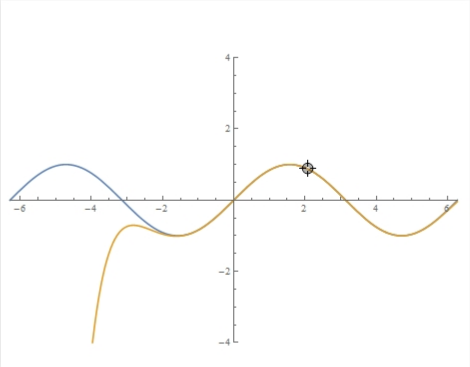
$$n = 8$$



Example 2: Taylor Series

Consider the sequence of Taylor polynomials centered at $x = 0.5$ approximating the function $f(x) = \sin x$.

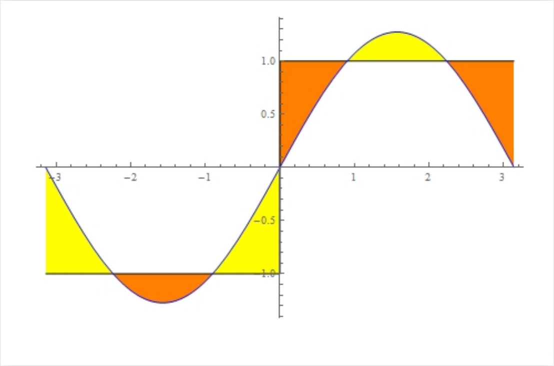
$$n = 11$$



Example 3: Fourier Series

Or, for something more exotic, consider the sequence of Fourier trigonometric polynomials approximating the step function $f(x) = \text{sgn}\, x$.

$$n = 1$$



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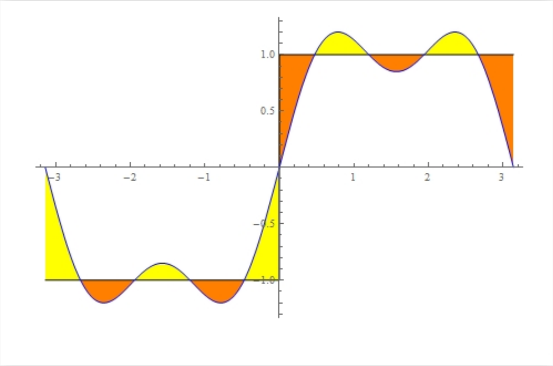
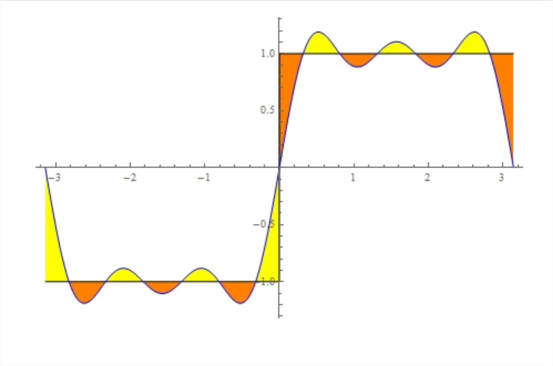
$$n = 4$$



Example 3: Fourier Series

Or, for something more exotic, consider the sequence of Fourier trigonometric polynomials approximating the step function $f(x) = \operatorname{sgn} x$.

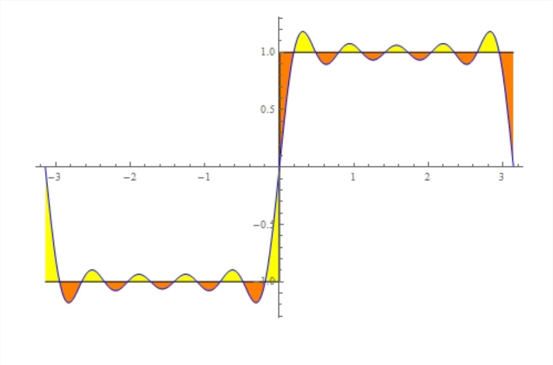
$$n = 6$$



Example 3: Fourier Series

Or, for something more exotic, consider the sequence of Fourier trigonometric polynomials approximating the step function $f(x) = \operatorname{sgn} x$.

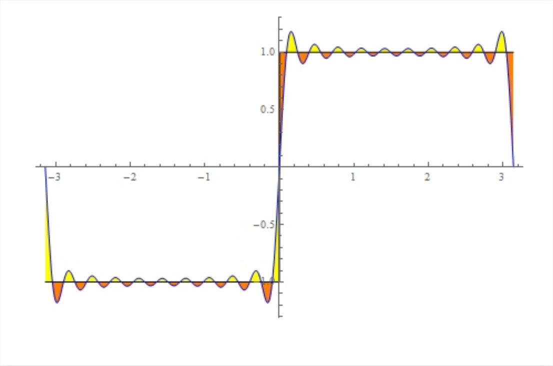
$$n = 10$$



Example 3: Fourier Series

Or, for something more exotic, consider the sequence of Fourier trigonometric polynomials approximating the step function $f(x) = \text{sgn}\, x$.

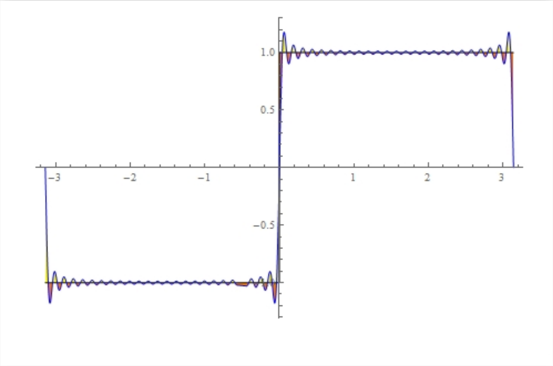
$$n = 20$$



Example 3: Fourier Series

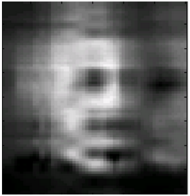
Or, for something more exotic, consider the sequence of Fourier trigonometric polynomials approximating the step function $f(x) = \text{sgn}\, x$.

$$n = 50$$



Example 4: Matrix Approximations

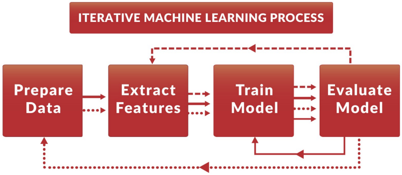
The matrix on the left is an approximation of the matrix on the right:



Example 5: Machine Learning

Machine learning works by:

- proposing an initial model M_0, and
- a way of refining the model M_k into model M_{k+1}.



Question: Does the sequence of models stabilize?

Note: Training a model often takes several weeks on many CPU or GPUs. It would be nice to know ahead of time that this process will converge.

Image: Dmitri Petrov

The Big Idea

Fréchet (1906): *The notions of convergence and approximation can be studied independent of the setting!*

This will be the approach of our course. It is the reason analysis is ubiquitous in mathematics.

Application 1: Formalization of Calculus

Convergence and approximation are the basis of calculus. Its methods were in use for centuries, sometimes with surprisingly incorrect results.

Goal: Make calculus precise.

Application 2: Compression

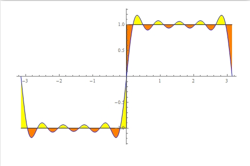
Idea: Find a sequence of“simple” objects which converge to the object we want to approximate, and truncate.

Suppose that we used the Fourier series to approximate the $sign$ function:

$$f(x) \approx \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \dots$$

As before, the first few terms get us:

$$n = 10$$



So instead of storing individual values of $f(x)$, one can just store the vector

$$\left(\frac{1}{2}, \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \dots, \frac{2}{19\pi}\right).$$

Application 3: Building models

Theorem: Under certain circumstances, a sequence of continuous functions is known to converge to another continuous function.

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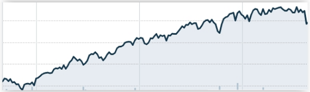
Note: Using this result N. Wiener constructed a model for the price for equities that is a continuous function.



Application 4: A new method of proof

Suppose that you would like to know that G has some property, but you can't prove it. But you know that you can approximate G by a sequence:

$$G_1, G_2, \ldots, G_n, \ldots$$

and the G_n are easier to deal with. The following is a powerful technique of proof:

Idea: Find a *simple* sequence that converges to G. Prove that each of the G_n have this property, and then conclude that G must also have this property.

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Note: Lots of details missing.

First Things First

Problem: Decide what it should mean for a sequence of objects to "converge?"

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The first thing we need to understand is **distance.** Following Fréchet, we'd like to do this in as much generality as we can.