Stone-Weierstrass: post-proof analysis

Thomas Pietraho

Introduction to Analysis

Why did Weierstrass do what he did?

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How did he know how to do it?

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Can one do it better?

First let's examine:

$$Q_n(x) = c_n(1 - x^2)^n.$$

A new definition

Recall: For a continuous function f, we defined

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)\,dt.$$

Definition: Suppose that f and g are continuous. Define a new function $f * g$ by

$$(f * g)(x) = \int_{-1}^{1} f(x+t)g(t)\,dt.$$

This is the *convolution* of f and g.

In this language, $P_n = f * Q_n$, and Q_n is the *kernel* of the convolution. Ultimately, we showed that $P_n \to f$.

A sequence of functions

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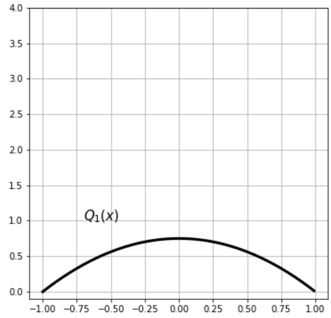

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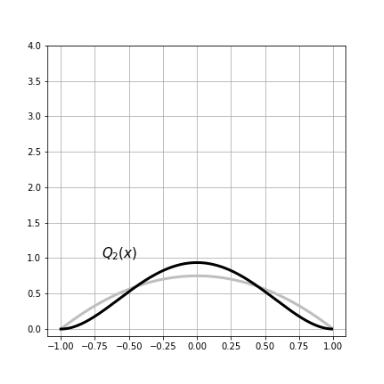

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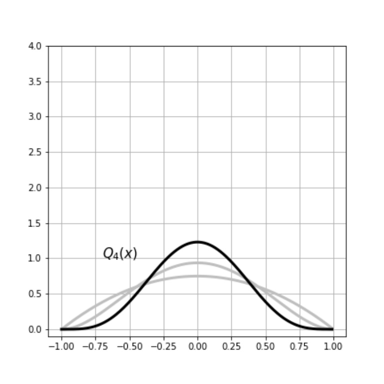

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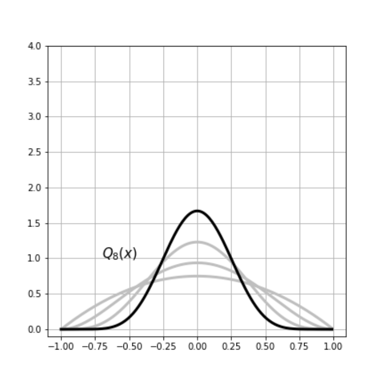

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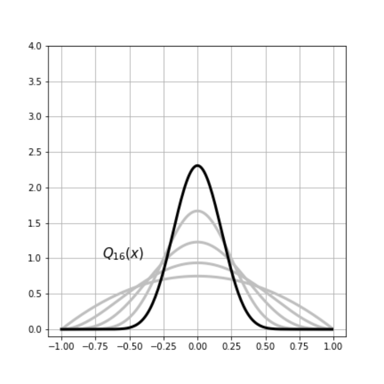

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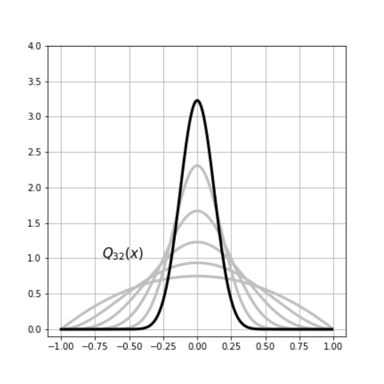

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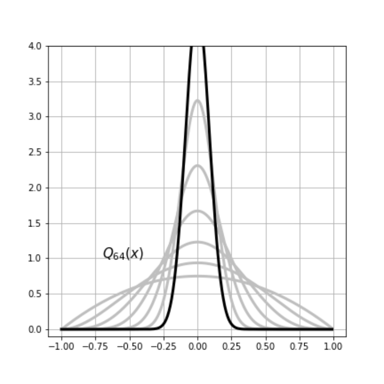

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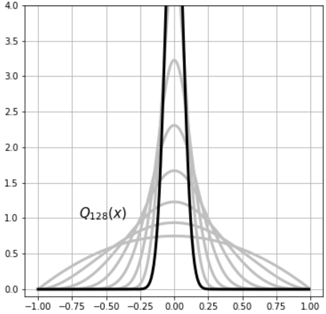

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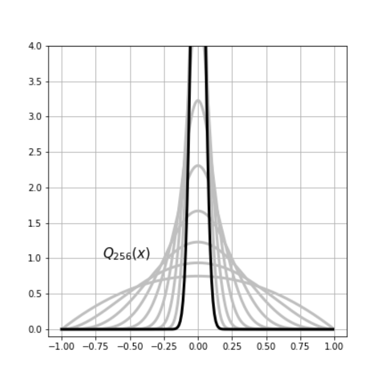

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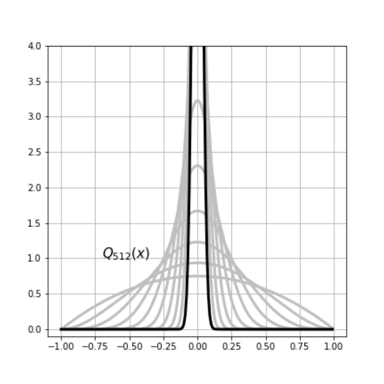


A sequence of functions

$$Q_n(x) = c_n(1 - x^2)^n$$

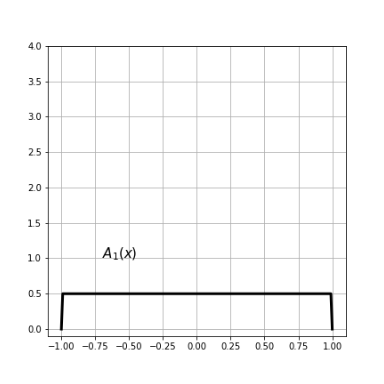
We used three properties of Q_n :

1. each $\int_{-1}^{1} Q_n = 1$
2. each Q_n is eventually small outside every interval $[-\delta, \delta]$
3. each Q_n is a polynomial.

We proved that $P_n \to f$ using (1) and (2). Item (3) allowed us to conclude that each P_n is a polynomial. There are many more sequences of functions that satisfy (1) and (2).

Another sequence of functions

$$A_n(x) = \frac{n}{2} \cdot \chi_{[-\frac{1}{n}, \frac{1}{n}]}$$



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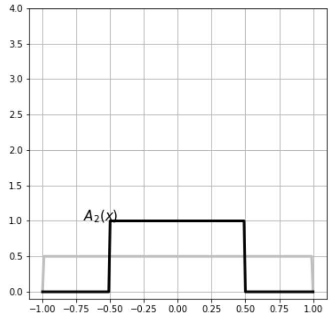

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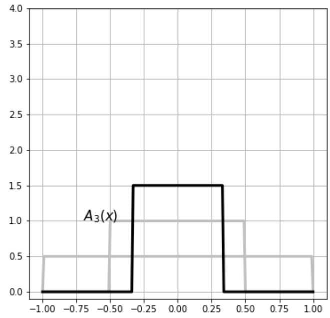

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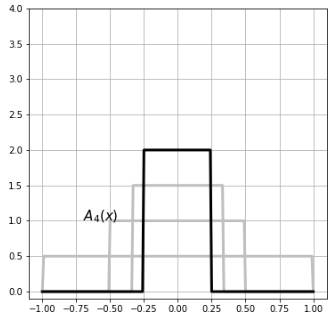

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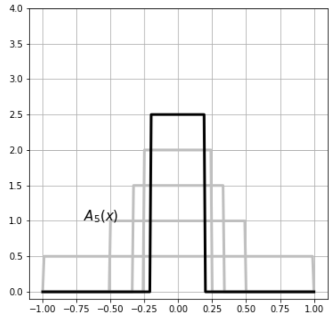

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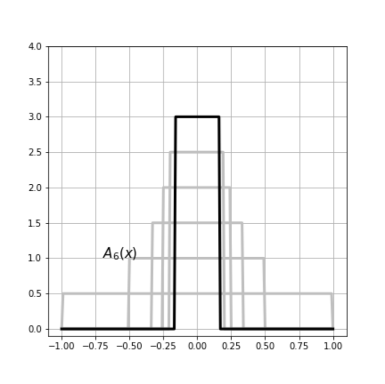

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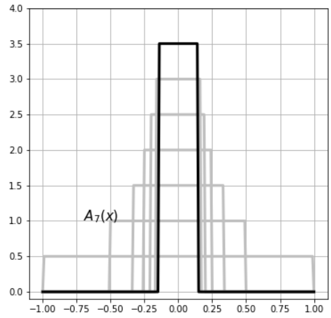

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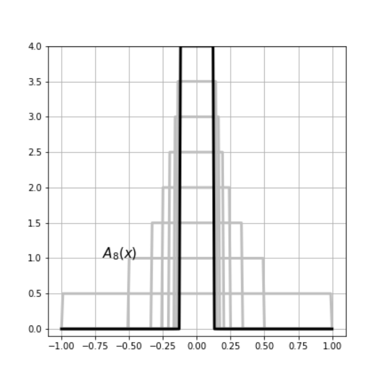
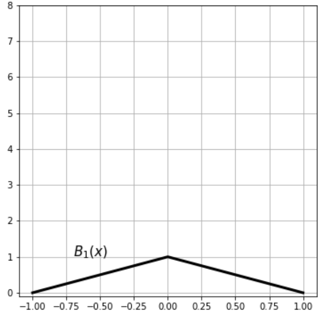


And another sequence of functions

$B_n(x)$ as below



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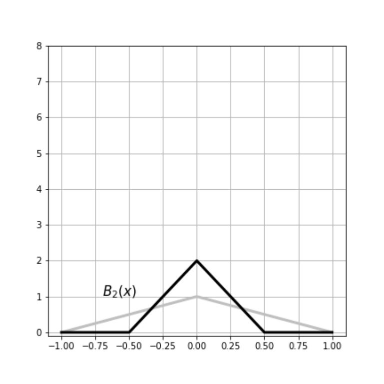

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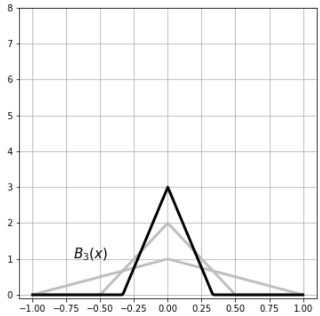

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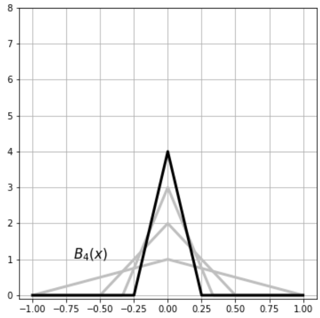

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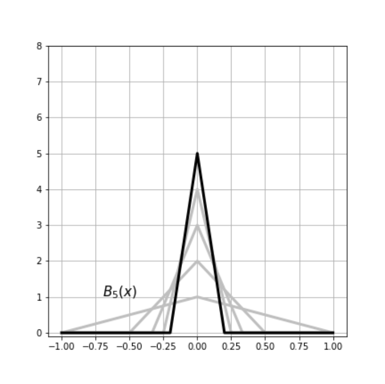

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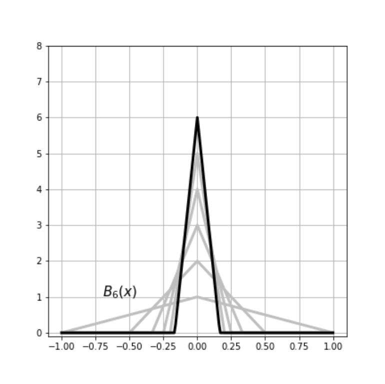

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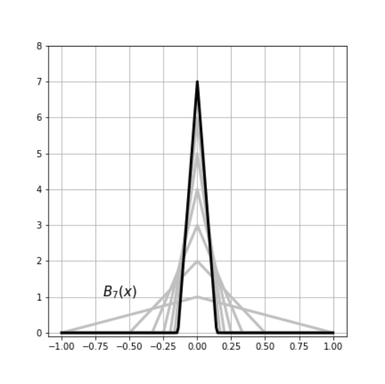

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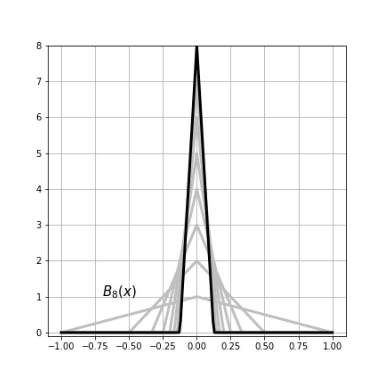

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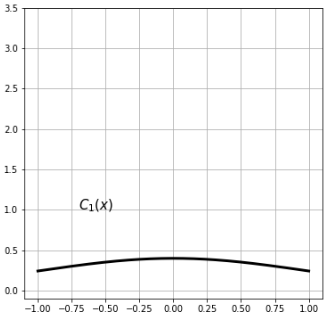

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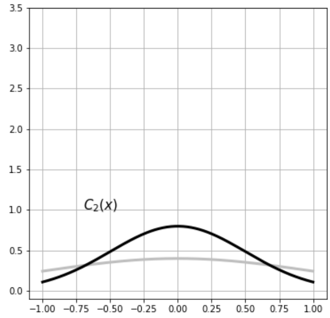

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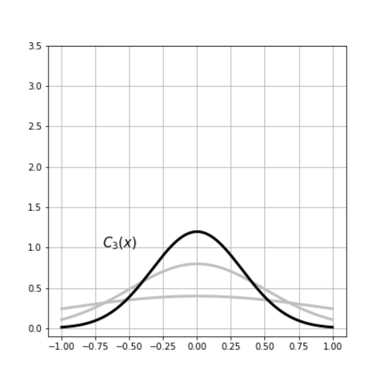

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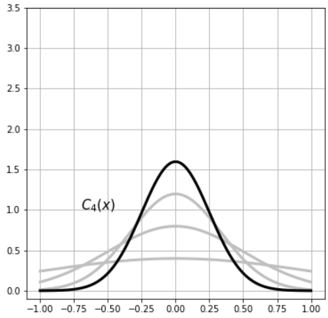

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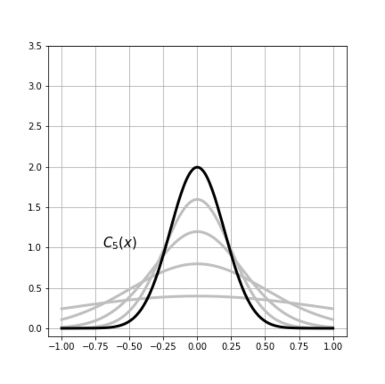

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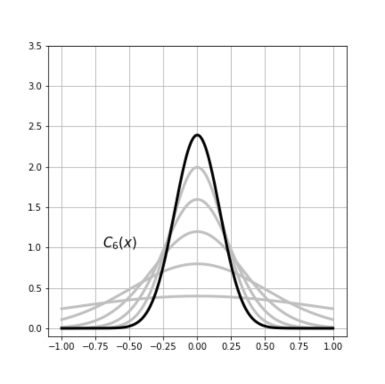

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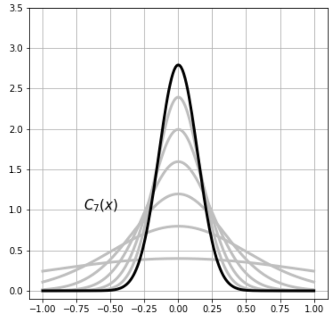

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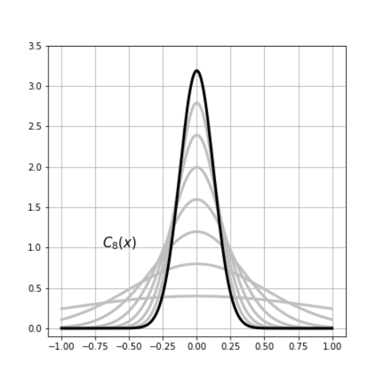

Meaning

The kernels A_n, B_n, C_n, and Q_n all satisfy properties (1) and (2):

1. each $\int_{-1}^{1} Q_n = 1$
2. each Q_n is eventually small outside every interval $[-\delta, \delta]$

Let's focus on the A_n:

$$f * A_n(x) = \int_{-1}^{1} f(x+t) A_n(t)\, dt = \int_{-\frac{1}{n}}^{\frac{1}{n}} f(x+t) \cdot \tfrac{n}{2}\, dt$$

$$= \frac{1}{\left(\frac{2}{n}\right)} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(x+t)\, dt = \frac{1}{\left(\frac{2}{n}\right)} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(s)\, ds$$

This is just the average value of f on the interval $[x - \frac{1}{n}, x + \frac{1}{n}]$!

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Meaning

$f * A_n$ is the average value of f on the interval $[x - \frac{1}{n}, x + \frac{1}{n}]$!

$f * A_n$ is a smoothed version of the function f.

It is not surprising that $P_n = f * A_n \to f$. As n grows, the average is taken over smaller intervals.

In general:

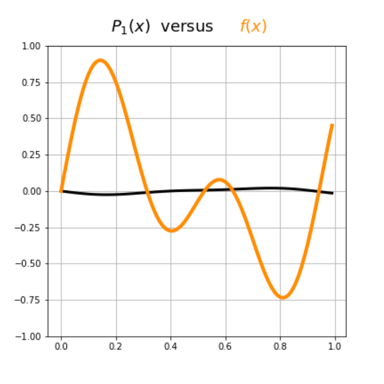
$f * g$ is a *weighted average* of f around x, where the function g determines the nature of the weighting.

$f * K_n \to f$ if K_n satisfies (1) and (2).

How well does this work?

$$f(x) = \cos(3x) * \sin(10 * x)$$

$$P_n(x) = f * Q_n$$



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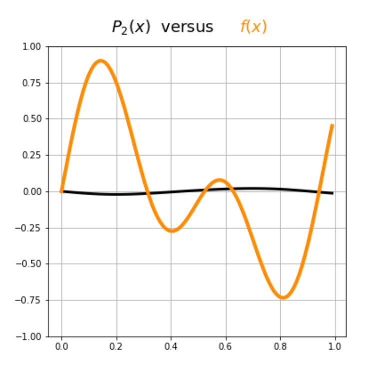

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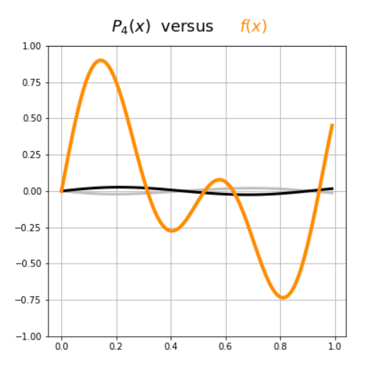

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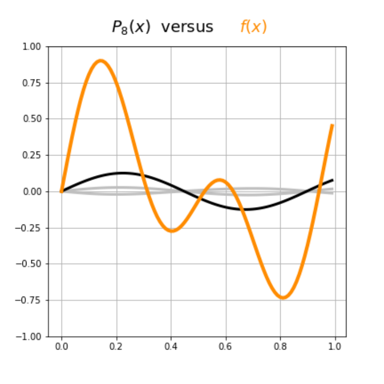

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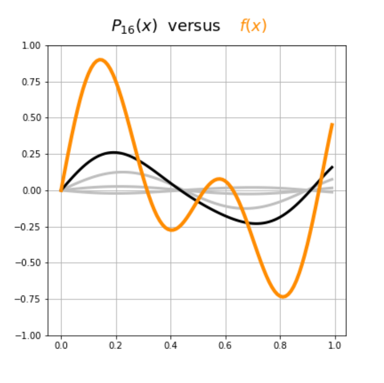

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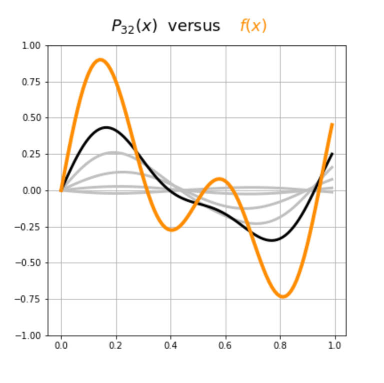

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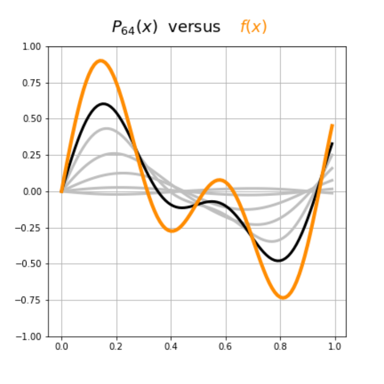

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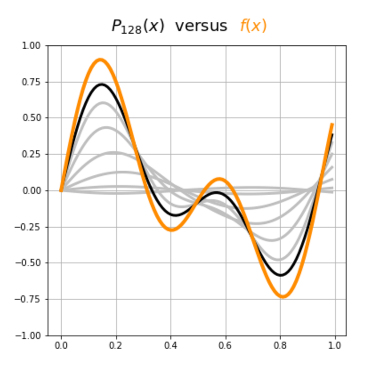

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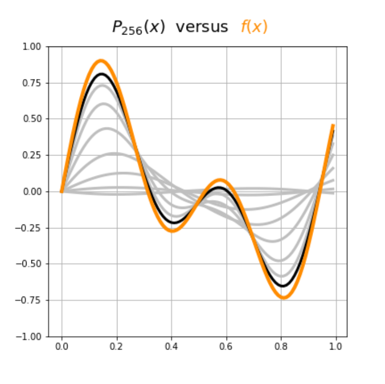

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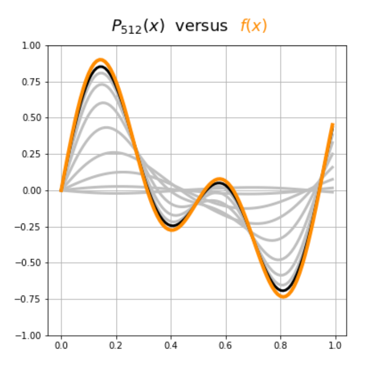

Problems

Find polynomial kernels R_n that give a faster rate of convergence for $f * R_n \to f$.

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Given a family of functions f, find kernels K_n for which convergence $f * K_n \to f$ is fast and the functions $f * K_n$ are simple from the perspective of the problem you are studying.