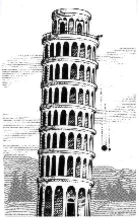
Continuity: Philosophy

Thomas Pietraho

Bowdoin College

Example 1

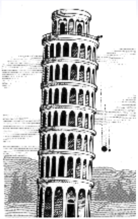


x is the time after a stone is dropped from a building and y is the number of meters that the stone travels after x seconds:

$$y = g\frac{x^2}{2}$$

Question: *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Example 1



x is the time after a stone is dropped from a building and y is the number of meters that the stone travels after x seconds:

$$y = g\frac{x^2}{2}$$

Question: *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Answer: *Yes!*

Example 2

Suppose that we have a rod with one end attached by a hinge to the ground in such a way that the rod drop freely to either side. The position of the rod is described by the angle it makes with the vertical, which can vary from -90 *(the rod is lying on the ground pointing to our left) through* 0 *(the rod is vertical) to* $+90$ *(the rod is on the ground pointing right).*

First, hold the rod still in some position, then let go. If the rod moves, wait until it stops and then record its position.

$$y = \begin{cases} -90 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 90 & \text{if } x > 0 \end{cases}$$

Question: *Given any desired level of accuracy for measuring* $y = f(x)$*, can we ensure that level of accuracy by measuring* x *sufficiently accurately?*

Example 2

Suppose that we have a rod with one end attached by a hinge to the ground in such a way that the rod drop freely to either side. The position of the rod is described by the angle it makes with the vertical, which can vary from -90 *(the rod is lying on the ground pointing to our left) through* 0 *(the rod is vertical) to* $+90$ *(the rod is on the ground pointing right).*

First, hold the rod still in some position, then let go. If the rod moves, wait until it stops and then record its position.

$$y = \begin{cases} -90 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 90 & \text{if } x > 0 \end{cases}$$

Question: *Given any desired level of accuracy for measuring* $y = f(x)$*, can we ensure that level of accuracy by measuring* x *sufficiently accurately?*

Answer: *No!*

Toward a definition

Question: *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Toward a definition

Question: *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

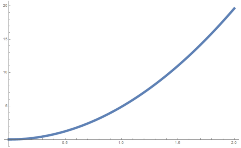
Idea: *Functions for which the answer to the above is YES should be called continuous, and those for which the answer is NO should be discontinuous.*

Toward a definition

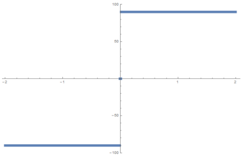
Question: *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Idea: *Functions for which the answer to the above is YES should be called continuous, and those for which the answer is NO should be discontinuous.*

This makes some sense in terms of our intuition for continuous functions:



(a) Example 1



(b) Example 2

Toward a definition

Question: (Version 1) *Given any desired level of accuracy for measuring* $y = f(x)$*, can we ensure that level of accuracy by measuring* x *sufficiently accurately?*

Toward a definition

Question: (Version 1) *Given any desired level of accuracy for measuring* $y = f(x)$*, can we ensure that level of accuracy by measuring* x *sufficiently accurately?*

Question: (Version 2) *Given any desired level of accuracy* $\epsilon > 0$ *for measuring* $y = f(x)$*, can we ensure that level of accuracy by measuring* x *sufficiently accurately, to within some number* $\delta > 0$*?*

Toward a definition

Question: (Version 1) *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Question: (Version 2) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately, to within some number $\delta > 0$?*

Question: (Version 3) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we say that if x is measured to within $\delta > 0$, we can be sure that $f(x)$ is predicted to within ϵ ?*

Toward a definition

Question: (Version 1) *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Question: (Version 2) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately, to within some number $\delta > 0$?*

Question: (Version 3) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we say that if x is measured to within $\delta > 0$, we can be sure that $f(x)$ is predicted to within ϵ ?*

Question: (Version 4) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we say that if our approximation x' to x is measured to within $\delta > 0$, we can be sure that our prediction $f(x')$ is within ϵ of $f(x)$?*

Toward a definition

Question: (Version 1) *Given any desired level of accuracy for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately?*

Question: (Version 2) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we ensure that level of accuracy by measuring x sufficiently accurately, to within some number $\delta > 0$?*

Question: (Version 3) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we say that if x is measured to within $\delta > 0$, we can be sure that $f(x)$ is predicted to within ϵ ?*

Question: (Version 4) *Given any desired level of accuracy $\epsilon > 0$ for measuring $y = f(x)$, can we say that if our approximation x' to x is measured to within $\delta > 0$, we can be sure that our prediction $f(x')$ is within ϵ of $f(x)$?*

Functions that satisfy this criterion will be called continuous. We are ready for a formal definition.