Inefficiency of Equilibria & Routing Games

Ref: Ch 17, 18 [AGT]

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Split or steal game

- NE outcome vs. socially best/optimal outcome

<table>
<thead>
<tr>
<th>Payoff matrix</th>
<th>Lucy</th>
<th>Tony</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Split</td>
<td>$33K, $33K</td>
</tr>
<tr>
<td>Split</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steal</td>
<td>$66K, $0+fr.</td>
<td>$0, $0</td>
</tr>
</tbody>
</table>
Prisoner's "dilemma" game

- Again: NE outcome vs socially optimal outcome

<table>
<thead>
<tr>
<th>Suspect 1</th>
<th>Not Confess</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Confess</td>
<td>1, 1</td>
<td>10, 0</td>
</tr>
<tr>
<td>Confess</td>
<td>0, 10</td>
<td>5, 5</td>
</tr>
</tbody>
</table>

Costs (negative of payoffs)

Measuring the inefficiency of NE

- What is the objective function to compare different outcomes?
  - Utilitarian
  - Egalitarian
- How to deal with multiplicity of NE?
  - Inefficiency of which NE?
  - Price of anarchy vs. price of stability
Price of Anarchy (PoA)

\[ \text{PoA} = \frac{\text{Worst objective function value among all NE}}{\text{Objective function value of optimal outcome}} \]

Price of Stability (PoS)

\[ \text{PoS} = \frac{\text{Best objective function value among all NE}}{\text{Objective function value of optimal outcome}} \]
### Example

- **Calculate PoA and PoS**

#### Suspect 1

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**Costs (negative of payoffs)**

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#### Column player

<table>
<thead>
<tr>
<th>Payoff matrix</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>21, -1</td>
<td>10, 0</td>
</tr>
<tr>
<td>D</td>
<td>100, 10</td>
<td>7, 8</td>
</tr>
</tbody>
</table>

**Costs (negative of payoffs)**
PoA vs. PoS

- Consider costs
- PoA and PoS will be $\geq 1$
- PoA = PoS when all NE have the same cost (e.g., unique NE)
- In general, PoA $\geq$ PoS

PoA vs. PoS

- PoA: worst case guarantee in a system of independent agents
- PoS: measures benefit of a protocol or proposed outcome
Pigou’s Example
PoA and PoS

Routing Games
**Model: nonatomic selfish routing**

- Multicommodity flow network
  - Directed network with multiple (source, sink) pairs
  - Each (source, sink) pair is called a commodity
  - $r_i$ amount of traffic for each commodity $i$
- Each edge $e$ has a delay or cost function $c_e$
  - Every car going through an edge gets same delay
- Cost of a path = sum of edge costs
  - Note: cost doesn't depend on identity of players
  - Congestion games

**Equilibrium flow**

- Let $f$ be a feasible flow (combining all commodities)
- $f$ is equilibrium flow if
  - All detours have higher (or equal) delay

![Diagram of a directed network with equilibrium flow](image)
More complex graphs

- Cost of the whole flow (all red and green)
- Total delay on each edge

Surprise!

- Price of anarchy for any nonatomic routing game with linear costs $\leq 4/3$
Example: nonlinear Pigou

- Consider large number $p$; 1 unit of traffic
- Equilibrium cost = ? Optimal cost = ? PoA = ?

\[ c(x) = 1 \]

\[ c(x) = x^p \]

PoA $\rightarrow +\infty$

Example: Braess' paradox

- 1 unit of traffic

\[ c(x) = x \quad c(x) = 1 \]

\[ c(x) = 1 \quad c(x) = x \]

PoA = ?
Braess' paradox

New super highway between v and w

PoA = 4/3