CSCI 3210: Computational Game Theory

Approximation Algorithms

Ref: Vazirani [Blackboard]

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Task

- Take an NP-Hard optimization problem
- Give an approximately optimal solution in poly time
  - Finding optimal is NP-Hard
  - Lower bound as a yardstick
**LP duality**

- **Primal LP**
  - Maximize ...

- **Dual LP**
  - Minimize ...

Increasing objective function

**Gap?**

**Vertex cover problem**

- Given a graph, select the minimum number of nodes such that at least one endpoint of every edge is selected.

- **Answer?**
Optimization

- $x_t$ is 1 if node $t$ is picked and 0 otherwise
- Is the following a usual linear program?

Minimize $\Sigma_t x_t$
Subject to

\[ x_u + x_v \geq 1, \text{ for each edge } (u, v) \]
\[ x_t \in \{0, 1\}, \text{ for each node } t \]

Integer linear program (ILP)
In general NP-hard

LP relaxation

Minimize $\Sigma_t x_t$
Subject to

\[ x_u + x_v \geq 1, \text{ for each edge } (u, v) \]
\[ x_t \geq 0, \text{ for each node } t \]
Our first approximation algorithm

- LP relaxation
- Rounding
- 2-approximation algorithm for vertex cover
  - Proof.

Primal-dual approximation (using relaxed LP)

- **Primal**
  - Minimize $\Sigma_t x_t$
  - Subject to
    - $x_u + x_v \geq 1$, for each edge $(u, v)$
    - $x_t \geq 0$, for each node $t$

- **Dual**
  - Maximize $\Sigma_e y_e$
  - Subject to
    - $\Sigma_{\text{all edges } e \text{ incident on } t} y_e \leq 1$, for each node $t$
    - $y_e \geq 0$, for each edge $e$

$c$ and $b$ are vectors of 1s. Maximum matching problem (relaxed)
Maximum matching problem

- Select the maximum number of edges so that the selected edges do not share any vertex.

- What's your solution?

Paradox?

- Vertex cover is NP-Hard
- Maximum matching is dual of vertex cover
- Maximum matching is polynomial-time solvable!
Answer: strong vs weak duality

Gap?

Maximum matching
ILP

Increasing objective function

Minimum vertex cover
ILP

Lower bounding technique for approx. algorithm

Gap?

Maximum matching
Maximal matching

Increasing objective function

Minimum vertex cover
Maximum vs. maximal matching

- Find a matching that’s maximal but not maximum

Approx. algorithm 2 for vertex cover & analysis
(written on board)
Set Cover Problem

- Given a universe $U$ of $n$ elements and a collection of subsets of $U$, $S = \{S_1, \ldots, S_m\}$, find the minimum # of subsets to cover all $n$ elements.
SET COVER

- Decision version is NP-complete

Greedy algorithm

1. \( C = \{ \} \)
2. While \( C \neq \text{Universe} \) do
   i. Pick a subset \( S' \) which covers the max # of new elements
   ii. \( C \leftarrow C \cup S' \)
3. Output the picked subsets

Cost: Cost of each picked subset is 1

Price: Each of the \( k \) new element covered by a picked subset pays a price of \( 1/k \)
LP

**Primal: ILP**

minimize $\sum_s x_s$

s.t.

$\sum_{S:e \in S} x_s \geq 1, \forall e \in U$

$x_s \in \{0, 1\}, \forall S \in S$

**Relaxed: Covering**

minimize $\sum_s x_s$

s.t.

$\sum_{S:e \in S} x_s \geq 0, \forall S \in S$

maximize $\sum_e y_e$

s.t.

$\sum_{e:e \in S} y_e \leq 1, \forall S \in S$

$y_e \geq 0, \forall e \in U$

**Dual: Packing**

Lower bound

**Increasing obj. func.**

(Min) Relaxed LP

ILP

OPT

OPT$_f$

(Integrality gap)

(Max) Dual LP

LB: Any feasible solution to dual
Finding a dual feasible soln

\[ \text{maximize } \sum_e y_e \]
\[ \text{s.t. } \sum_{e : e \in S} y_e \leq 1, \forall S \in \mathcal{S} \]
\[ y_e \geq 0, \forall e \in U \]

- Claim. \( y_e = \text{price}(e)/H_n \) is a feasible soln.
  - \( H_n = 1 + 1/2 + \ldots + 1/n \)
  - Proof. Show that no subset \( S \) is overpacked.

Main result

- Theorem. Greedy set cover is an \( O(\ln n) \) approximation.
- Proof.

\[ \text{Size of set cover} = \sum_e \text{price}(e) \]
\[ = \sum_e (H_n \times y_e) \]
\[ = H_n \sum_e y_e \]
\[ \leq H_n \times \text{OPT}_f \]
\[ \leq H_n \times \text{OPT} \]

\( (\text{Min}) \text{ Relax LP} \)
\( (\text{Max}) \text{ Dual LP} \)
\( \text{ILP} \)
\( \text{OPT} \)
\( \text{OPT}_f \)

Tight example?
Applications

- Geometric coverage problems
  - Guarding problems
  - Facility location problems

- Crew scheduling problems
  - [Yelbay et al., 2014]: American Airlines, bus driver, etc.
  - [Pezzella & Faggioli, 1997]: Italian Railways Company

- Testing programs: smartly choosing test cases
- Clustering and outlier detection

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<th>Number of post offices required</th>
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<tr>
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Source: International Postal Corporation (2020)