CSCI 3210: Computational Game Theory

Brief Intro to Game Theory

Mohammad T. Irfan

Email: mirfan@bowdoin.edu
Course Website: www.bowdoin.edu/~mirfan/CSCI-3210.html
Office Hours: Tu & W 2:30–5 (VAC 309)

Syllabus and required background

- Course website
  - www.bowdoin.edu/~mirfan/CSCI-3210.html
Games and game theory: A brief introduction

Reading: Ch. 1 of AGT Book

Game Theory

- “Game”
  - Ernst Zermelo (1913): In any chess game that does not end in a draw, a player has a winning strategy

- Mathematical theory of strategic decision making
  - John von Neumann (1944)
Example: Split or Steal

- [https://www.youtube.com/watch?v=yM38mRHY150](https://www.youtube.com/watch?v=yM38mRHY150)
- Rules of the game
- Outcome

One possible model

<table>
<thead>
<tr>
<th></th>
<th>Lucy</th>
<th>Tony</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Payoff matrix</strong></td>
<td><strong>Split</strong></td>
<td><strong>Steal</strong></td>
</tr>
<tr>
<td><strong>Split</strong></td>
<td>$33K, $33K</td>
<td>$0+fr., $66K</td>
</tr>
<tr>
<td><strong>Steal</strong></td>
<td>$66K, $0+fr.</td>
<td>$0, $0</td>
</tr>
</tbody>
</table>

What will happen?
Why did they end up with 0?

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Nash Equilibrium
Everyone plays his/her best response to others simultaneously.

Nash equilibrium

Practical scenarios
= Stable outcome
= Nash equilibrium
Applications

- Application: market equilibria
  - Predict where the market is heading to
- Mechanism design and auctions
  - Google and Yahoo apply game-theoretic techniques
    - Keyword search auction
    - Spectrum allocation among wireless companies

Applications

- Understanding the Internet
  - Selfish routing is a constant-factor off from optimal
Other applications

- Load balancing and resource allocation
- p2p and file sharing systems
- Cryptography and security
- Social and economic networks, etc.

Next:
Formal discussion
Game

- One-shot games (simultaneous move)
- 3 components
  - Players
  - Strategies/actions
  - Payoffs

Payoff matrix

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<td>$0, $0</td>
</tr>
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Nash equilibrium (NE)

- A joint strategy (one strategy/player) s.t. every player plays his/her best response to others simultaneously
- (Equiv.) A joint strategy s.t. no player gains by deviating unilaterally
  - Useful for checking whether a cell is NE
Famous example: prisoner's dilemma

- What will they do?

<table>
<thead>
<tr>
<th>Suspect 1</th>
<th>Not Confess</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Confess</td>
<td>-1, -1</td>
<td>-10, 0</td>
</tr>
<tr>
<td>Confess</td>
<td>0, -10</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

Assumptions

- Payoffs reflect player’s preference
- Payoffs are known to all
- Actions are known to all (different players could have different actions - but everyone knows everyone’s actions)
- Each player wants to maximize own payoff subject to others' actions
Commonly Used Terms

Best response

- Best strategy of a player, given the other players' strategies
- Always exists!
(Strictly) dominant strategy

- A strategy of a player that is (strictly) better than any of his other strategies, no matter what the other players do
- Does not always exist

Pure-strategy Nash equilibrium

- Players do not use any probability in choosing strategies as they do in "mixed-strategy" (to be covered later)
  - Every player plays his/her best response to others simultaneously
Quiz

- What is the difference between a dominant strategy and a best response?

Quiz

- Here’s a clip from the movie *A Beautiful Mind* that tries to portray John Nash’s discovery of Nash equilibrium.
- Is this actually a Nash equilibrium?
Misconceptions

- Equilibrium signifies a tie/draw/balance
- Equilibrium outcome is the best possible outcome for all players (*A Beautiful Mind*)
- Self-interested players want to hurt each other

Quick review

<table>
<thead>
<tr>
<th>Payoff matrix</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>10, 50</td>
<td>5, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, 0</td>
<td>5, 10</td>
</tr>
</tbody>
</table>

(Weakly) dominant strategy

Column Player

Pure-strategy Nash equilibria

Neither is a dominant strategy

The concept of Nash equilibrium doesn't say how an equilibrium happens!
Questions

- Does NE always exist? (Answer later …)
- If it exists, is it unique?

Games with multiple NE

1. Battle of the sexes (Coordination)
2. Hawk-dove game (anti-coordination)
Hawk-dove game in a population

“a hawk will do very well in a population consisting of doves” [David Easley (2010)]

Does NE always exist?

Mixed-strategy NE

1. Normandy Landing
2. Matching pennies game
Pure-strategy vs. mixed-strategy NE

Hawk-dove game
Trump-Nieto game

More on mixed-strategy NE

Penalty kick game
*What does playing mixed strategy mean?*
Penalty kick game

Penalty kick game (continued)
Penalty kick game (continued)

Penalty kick game - first model

- \( E[\text{GK plays Left}] \)
  
  \[
  = p(1) + (1-p)(-1) \\
  = 2p - 1 
  \]

- \( E[\text{GK plays Right}] \)
  
  \[
  = p(-1) + (1-p)(1) \\
  = 1 - 2p 
  \]

- \( 2p - 1 = 1 - 2p \) \( \Rightarrow \) \( p = \frac{1}{2} \)

- Similarly, \( q = \frac{1}{2} \)

Zero-sum Game

<table>
<thead>
<tr>
<th>Goalkeeper</th>
<th>Left (q)</th>
<th>Right (1-q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shooter Left (p)</td>
<td>-1, +1</td>
<td>+1, -1</td>
</tr>
<tr>
<td>Shooter Right (1-p)</td>
<td>+1, -1</td>
<td>-1, +1</td>
</tr>
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</table>
Penalty kick game (real-world)

- "Professionals Play Minimax" - Ignacio Palacios-Huerta

Equilibrium probabilities (computed by solving equations) match real-world probabilities from data!

<table>
<thead>
<tr>
<th></th>
<th>Left (0.38)</th>
<th>Right (0.62)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>0.58, 0.42</td>
<td>0.95, 0.05</td>
</tr>
<tr>
<td>Right</td>
<td>0.93, 0.07</td>
<td>0.70, 0.30</td>
</tr>
</tbody>
</table>

From real-world data

What does mixed strategy mean?

- Active randomization - tennis, soccer
- Proportion interpretation - evolutionary biology
- Probabilities of player 1 are the beliefs of player 2 about what player 1 is doing (Bob Aumann)
- Misconception
  - Players just choose probabilities
- Correct
  - players play pure strategies chosen according to these probabilities
Von Neumann’s Theorem (1928)

- Every finite 2-person zero-sum game has a mixed equilibrium

John von Neumann (1903 - 1957)

Theorem of Nash (1950)

- Every finite game has an equilibrium in mixed strategies
- Reading

John F. Nash (1928 - 2015)
Nobel Prize, 1994
Other solution concepts

- Socially optimal solution
  - Joint strategy that maximizes the sum of payoffs
  - The sum of payoffs is called social welfare

- Pareto-optimal solution
  - A joint-strategy is NOT Pareto-optimal if it is “dominated by” another joint-strategy such that:
    1. everyone’s payoff in the latter joint-strategy is at least as high as in the former, and
    2. at least one player gets strictly higher payoff in the latter.

- None of the above might be NE (Extremely rarely are they NE)

Key take-away messages

- Players act simultaneously, but NE outcome is stable in the sense that there is no incentive for unilateral deviation.
- There is always at least one mixed-strategy NE. A pure-strategy NE is not guaranteed.
- The concept of NE doesn't say how NE happens.
- NE is not a balance or tie. It is often times a socially-inefficient outcome.